A simplistic Sagnac ring

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1 The Sagnac ring

Two continuous monochromatic light beams are sent in opposite directions around a circular loop of radius r. The light beams are emitted from a common source, and are therefore in phase at the emitter. The light beams will meet each other back at the source, and the phase difference between them is measured. The ring is rotating around its centre with a peripheral speed v.

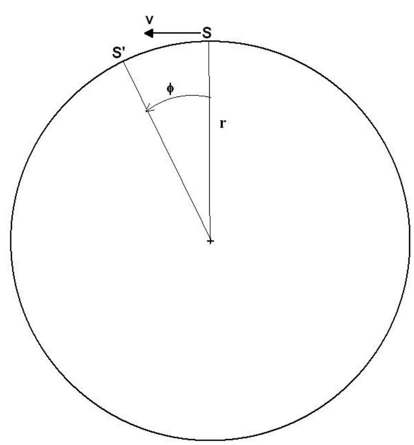

Figure 1: *The Sagnac ring*

We will calculate what the Special Theory of Relativity and emission theories predict the phase difference will be when the interferometer is rotating. By 'emission theory' we will mean a theory where the speed of light in vacuum is c relative to the source, and velocities transform according to the Galilean transform.

There are two ways of calculating this phase difference:

1. The difference in transit time Δt for the two beams can be calculated.
   The transit time is the time a plane of equal phase uses to move around the ring.
   The phase difference is then $\boldsymbol{\Delta\varphi = 2\pi\nu\,\Delta t}$ where $\boldsymbol{\nu}$ is the frequency of the light.

2. The number of wavelengths in the two beams can be compared.
   The phase difference is then $\boldsymbol{\Delta\varphi = 2\pi\,\Delta N}$ where $\boldsymbol{\Delta N}$ is the difference in the number of wavelengths in the two beams.

Even if the two methods necessarily must give the same result, we will calculate the predictions both ways.

2 The prediction of the Special Theory of Relativity

2.1 The difference in transit times method

The speed of light measured in the non-rotating inertial frame is, according to the Special Theory of Relativity, c for both beams.

The time t_f to go around the circular loop with radius r is for the beam going with the rotation:

$$(2\pi + \phi)\, r = c\, t_f \tag{1}$$

$$\phi = \frac{v}{r}\, t_f \tag{2}$$

$$t_f = \frac{2\pi r}{c - v} \tag{3}$$

The time t_b to go around the circular loop with radius r is for the beam going in the opposite direction:

$$(2\pi - \phi)\, r = c\, t_b \tag{4}$$

$$\phi = \frac{v}{r}\, t_b \tag{5}$$

$$t_b = \frac{2\pi r}{c + v} \tag{6}$$

The difference in the transit times Δt is:

$$\Delta t = t_f - t_b = \frac{2\pi r}{c - v} - \frac{2\pi r}{c + v} = \frac{4\pi r v}{c^2 - v^2} \tag{7}$$

Δt is the transit time measured in the non-rotating inertial frame where the centre of the ring is stationary. The phase detector is co-located with the source, and is moving along with it at the speed v. So the transit time as measured by the moving source/detector, will be:

$$\Delta t' = \Delta t \sqrt{1 - \left(\frac{v}{c}\right)^2} = \frac{4\pi r v}{c^2 - v^2} \sqrt{1 - \left(\frac{v}{c}\right)^2} = \frac{4\pi r v}{c^2 \sqrt{1 - \left(\frac{v}{c}\right)^2}} \tag{8}$$

The phase difference as measured by the moving phase detector will thus be:

$$\Delta\varphi = 2\pi\nu\Delta t' = \frac{2\pi c\, \Delta t'}{\lambda} = \frac{8\pi^2 r v}{\lambda c \sqrt{1 - \left(\frac{v}{c}\right)^2}} \tag{9}$$

where λ is the wavelength of the light in vacuum.

Ignoring second order terms in v/c, and inserting the area A of the ring and its angular velocity $\omega = v/r$, yields:

$$\Delta\varphi \simeq \frac{8\pi A\, \omega}{\lambda c} \tag{10}$$

This is in accordance with the experimentally verified equation for a Sagnac ring.

2.2 The difference in number of wavelengths method

Since the source is moving, the wavelength of the light in the beam going with the rotation will be Doppler shifted in the inertial frame where the centre of the ring is stationary:

$$\lambda_f = \frac{\sqrt{1-\left(\frac{v}{c}\right)^2}}{1+\frac{v}{c}}\lambda = \frac{\lambda c\sqrt{1-\left(\frac{v}{c}\right)^2}}{c+v} \tag{11}$$

Equivalently will the wavelength of the light in the beam going in the opposite direction be Doppler shifted:

$$\lambda_b = \frac{\sqrt{1-\left(\frac{v}{c}\right)^2}}{1-\frac{v}{c}}\lambda = \frac{\lambda c\sqrt{1-\left(\frac{v}{c}\right)^2}}{c-v} \tag{12}$$

The number of wavelengths in the beams will be:

$$N_f = \frac{2\pi r}{\lambda_f} = \frac{2\pi r\,(c+v)}{\lambda c\sqrt{1-\left(\frac{v}{c}\right)^2}} \tag{13}$$

$$N_b = \frac{2\pi r}{\lambda_b} = \frac{2\pi r\,(c-v)}{\lambda c\sqrt{1-\left(\frac{v}{c}\right)^2}} \tag{14}$$

The difference in the number of wavelengths will be:

$$\Delta N = N_f - N_b = \frac{2\pi r\,(c+v)}{\lambda c\sqrt{1-\left(\frac{v}{c}\right)^2}} - \frac{2\pi r\,(c-v)}{\lambda c\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{4\pi r v}{\lambda c\sqrt{1-\left(\frac{v}{c}\right)^2}} \tag{15}$$

The phase difference between the beams will thus be:

$$\Delta\varphi = 2\pi\Delta N = \frac{8\pi^2 r v}{\lambda c\sqrt{1-\left(\frac{v}{c}\right)^2}} \tag{16}$$

This phase difference is identical to the result we got in the previous section. So both methods give the same result, as they must.

Ignoring second order terms in v/c, and inserting the area A of the loop and its angular velocity $\omega = v/r$, yields:

$$\Delta\varphi \simeq \frac{8\pi A\,\omega}{\lambda c} \tag{17}$$

3 The prediction of emission theories

3.1 The difference in transit times method

The speed of light measured in the non-rotating inertial frame is, according to emission theories, $(c + v)$ for the beam going with the rotation, and $(c - v)$ for the beam going in the opposite direction.

The time t_f to go around the circular loop with radius r is for the beam going with the rotation:

$$(2\pi + \phi)\, r = (c + v)\, t_f \tag{18}$$

$$\phi = \frac{v}{r}\, t_f \tag{19}$$

$$t_f = \frac{2\pi r}{c} \tag{20}$$

The time t_b to go around the circular loop with radius r is for the beam going in the opposite direction:

$$(2\pi - \phi)\, r = (c - v)\, t_f \tag{21}$$

$$\phi = \frac{v}{r}\, t_b \tag{22}$$

$$t_b = \frac{2\pi r}{c} \tag{23}$$

The difference in the transit times Δt is:

$$\Delta t = t_f - t_b = \frac{2\pi r}{c} - \frac{2\pi r}{c} = 0 \tag{24}$$

The predicted phase difference is thus:

$$\Delta\varphi = 2\pi\nu\Delta t = 0 \tag{25}$$

3.2 The difference in number of wavelengths method

According to emission theories, wavelengths are not Doppler shifted. (Because of the Galilean transform.)

$$\lambda_f = \lambda_b = \lambda \tag{26}$$

The number of wavelengths in the beams will be:

$$N_f = N_b = \frac{2\pi r}{\lambda} \tag{27}$$

The difference in the number of wavelengths will be:

$$\Delta N = N_f - N_b = 0 \tag{28}$$

The predicted phase difference is thus:

$$\Delta\varphi = 2\pi\Delta N = 0 \tag{29}$$

4 Conclusion

We have shown that the Special Theory of Relativity predicts that the phase difference between the contra-moving light beams is:

$$\Delta\varphi = \frac{8\pi A\,\omega}{\lambda c\sqrt{1-\left(\frac{\omega r}{c}\right)^2}} \tag{30}$$

which within any practically possible precision of measurement is consistent with the experimentally verified equation:

$$\Delta\varphi = \frac{8\pi A\,\omega}{\lambda c} \tag{31}$$

The Sagnac experiment confirms the Special Theory of Relativity.

We have shown that emission theories predict no phase difference when the Sagnac ring is rotating. Since the experimentally verified equation for a Sagnac ring is $\Delta\varphi = \frac{8\pi A\,\omega}{\lambda c}$, the Sagnac experiment falsifies emission theories.