What the Special Theory of Relativity predicts for Fibre-Optic Gyroscopes

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1 Introduction

A fibre-optic gyroscope (FOG) is basically a coil of optic fibre where the light from a monochromatic light source is injected via a beam splitter into both ends of the fibre, such that there are two light beams going in opposite direction within the fibre. When the light beams are coming out of the ends of the fibre, they are led into a phase comparator which measure the phase difference between the light in the two beams. Due to the Sagnac effect, this phase difference will vary proportionally to the angular velocity of the FOG.

The commonly used equation for this phase difference is:

$$\Delta\varphi = \frac{8\pi N A\,\omega}{\lambda c} \tag{1}$$

where N is the number of turns in the coil, A is the area enclosed by the coil, λ is the wavelength of the light from the source in vacuum, c is the speed of light in vacuum, and ω is the angular velocity with which the FOG is rotating.

This equation is experimentally very well confirmed.

It is rather remarkable that according to this equation, the phase difference does not depend on the index of refraction n of the fibre. Intuitively one might expect that since the speed of light in the fibre is inversely proportional to n, the phase difference would show a similar dependency on n. Experimental evidence show that this is not the case.

We will calculate what the Special Theory of Relativity predicts the phase difference will be between the contra moving light beams in a rotating FOG, and we will see that according to this theory, the phase difference is indeed independent of the index of refraction in the fibre.

2 Calculation of the phase difference between the two beams

2.1 Methods

There are two ways of calculating this phase difference:

1. The difference in transit time Δt for the two beams can be calculated.
   The transit time is the time a plane of equal phase uses to move around the ring.
   The phase difference is then $\boldsymbol{\Delta\varphi = 2\pi\nu\,\Delta t}$ where $\boldsymbol{\nu}$ is the frequency of the light.

2. The number of wavelengths in the two beams can be compared.
   The phase difference is then $\boldsymbol{\Delta\varphi = 2\pi\,\Delta N}$ where $\boldsymbol{\Delta N}$ is the difference in the number of wavelengths in the two beams.

Even if the two methods necessarily must give the same result, we will calculate the predictions both ways.

2.2 The Fibre Optic Gyro

We will assume that our idealized FOG has the following parameters:

- Let the fibre be a single circular loop with radius r.
- Let the peripheral speed of the fibre be v as measured in the inertial frame where the centre of the ring is stationary.
- Let n be the index of refraction in the fibre.
- Let c be the speed of light in vacuum.

2.3 The prediction calculated with the difference in transit times

At any point on the fibre, the speed of light will be c/n as measured in an instantly co-moving inertial frame. We will transform this speed to the non-rotating inertial frame where the fibre is moving at the speed v, using the well known formula which follows from the Lorentz transform:

The beam going with the rotation:

$$c_f = \frac{\frac{c}{n} + v}{1 + \frac{\frac{c}{n} v}{c^2}} = \frac{c^2 + nvc}{nc + v} \tag{2}$$

The beam going in the opposite direction:

$$c_b = \frac{\frac{c}{n} - v}{1 - \frac{\frac{c}{n} v}{c^2}} = \frac{c^2 - nvc}{nc - v} \tag{3}$$

The time t_f to go around the circular loop with radius r is for the beam going with the rotation:

$$t_f\, c_f = 2\pi r + v\, t_f \tag{4}$$

$$t_f = \frac{2\pi r}{c_f - v} = \frac{2\pi r}{\frac{c^2 + nvc}{nc+v} - v} = 2\pi r \left(\frac{nc + v}{c^2 - v^2} \right) \tag{5}$$

The time t_b to go around the circular loop with radius r is for the beam going in the opposite direction:

$$t_b\, c_b = 2\pi r - v\, t_b \tag{6}$$

$$t_b = \frac{2\pi r}{c_b + v} = \frac{2\pi r}{\frac{c^2 - nvc}{nc - v} + v} = 2\pi r \left(\frac{nc - v}{c^2 - v^2} \right) \tag{7}$$

The difference in the transit times Δt is:

$$\Delta t = t_f - t_b = 2\pi r \left(\frac{nc + v}{c^2 - v^2} - \frac{nc - v}{c^2 - v^2} \right) = \frac{4\pi r v}{c^2 - v^2} \tag{8}$$

Note the rather remarkable result that although the transit times are approximately proportional to the index of refraction n, the difference in transit times is not affected by n at all.

Δt is the transit time measured in the non-rotating inertial frame where the centre of the loop is stationary. The phase detector is co-located with the source, and is moving along with it at the speed v. So the transit time as measured by the moving source/detector, will be:

$$\Delta t' = \Delta t \sqrt{1 - \left(\frac{v}{c} \right)^2} = \frac{4\pi r v}{c^2 - v^2} \sqrt{1 - \left(\frac{v}{c} \right)^2} = \frac{4\pi r v}{c^2 \sqrt{1 - \left(\frac{v}{c} \right)^2}} \tag{9}$$

The phase difference as measured by the moving phase detector will thus be:

$$\Delta \varphi = 2\pi \nu \Delta t' = \frac{2\pi c \, \Delta t'}{\lambda} = \frac{8\pi^2 r v}{\lambda c \sqrt{1 - \left(\frac{v}{c} \right)^2}} \tag{10}$$

where λ is the wavelength of the light in vacuum.

Ignoring second order terms in v/c, and inserting the area A of the loop and its angular velocity $\omega = v/r$, yields:

$$\Delta \varphi \simeq \frac{8\pi A \, \omega}{\lambda c} \tag{11}$$

This is consistent with the experimentally verified equation (1) for a FOG.

2.4 The prediction calculated by comparing number of wavelengths

- Let ν be the frequency of the light source as measured in an instantly co-moving inertial frame.
- Let λ be the wavelength in vacuum of light with frequency ν, $\lambda = c/\nu$.
- Let ν_f and λ_f be the frequency and wavelength of the beam that is moving with the rotation, as measured in the inertial frame where the centre of the ring is stationary.
- Let ν_b and λ_b be the frequency and wavelength of the beam that is moving in the opposite direction, as measured in the inertial frame where the centre of the ring is stationary.

Since the source is moving, the frequency of the beam going with the rotation will be Doppler shifted in the inertial frame where the centre of the ring is stationary:

$$\nu_f = \frac{\sqrt{1 - \left(\frac{v}{c} \right)^2}}{1 - \frac{v}{c_f}} \nu = \frac{c_f \sqrt{1 - \left(\frac{v}{c} \right)^2}}{c_f - v} \frac{c}{\lambda} \tag{12}$$

We have c_f from equation (2), and get:

$$\nu_f = \frac{\left(\frac{c^2 + nvc}{nc + v} \right) \sqrt{1 - \left(\frac{v}{c} \right)^2}}{\left(\frac{c^2 + nvc}{nc + v} \right) - v} \frac{c}{\lambda} = \frac{c + nv}{\lambda \sqrt{1 - \left(\frac{v}{c} \right)^2}} \tag{13}$$

The wavelength of the light in the beam going with the rotation will then be:

$$\lambda_f = \frac{c_f}{\nu_f} = \frac{\left(\frac{c^2+nvc}{nc+v}\right)}{\frac{c+nv}{\lambda\sqrt{1-\left(\frac{v}{c}\right)^2}}} = \frac{c\sqrt{1-\left(\frac{v}{c}\right)^2}}{nc+v}\,\lambda \tag{14}$$

Equivalently will the beam going in the opposite direction be Doppler shifted:

$$\nu_b = \frac{\sqrt{1-\left(\frac{v}{c}\right)^2}}{1+\frac{v}{c_b}}\,\nu = \frac{c_b\sqrt{1-\left(\frac{v}{c}\right)^2}}{c_b+v}\,\frac{c}{\lambda} \tag{15}$$

We have c_b from equation (3), and get:

$$\nu_b = \frac{\left(\frac{c^2-nvc}{nc-v}\right)\sqrt{1-\left(\frac{v}{c}\right)^2}}{\left(\frac{c^2-nvc}{nc-v}\right)+v}\,\frac{c}{\lambda} = \frac{c-nv}{\lambda\sqrt{1-\left(\frac{v}{c}\right)^2}} \tag{16}$$

The wavelength of the light in the beam going in the opposite direction will then be:

$$\lambda_b = \frac{c_b}{\nu_b} = \frac{\left(\frac{c^2-nvc}{nc-v}\right)}{\frac{c-nv}{\lambda\sqrt{1-\left(\frac{v}{c}\right)^2}}} = \frac{c\sqrt{1-\left(\frac{v}{c}\right)^2}}{nc-v}\,\lambda \tag{17}$$

The number of wavelengths in the beams will be:

$$N_f = \frac{2\pi r}{\lambda_f} = \frac{2\pi r\,(nc+v)}{\lambda c\sqrt{1-\left(\frac{v}{c}\right)^2}} \tag{18}$$

$$N_b = \frac{2\pi r}{\lambda_b} = \frac{2\pi r\,(nc-v)}{\lambda c\sqrt{1-\left(\frac{v}{c}\right)^2}} \tag{19}$$

The difference in the number of wavelengths will be:

$$\Delta N = N_f - N_b = \frac{4\pi r v}{\lambda c\sqrt{1-\left(\frac{v}{c}\right)^2}} \tag{20}$$

Note the rather remarkable result that although the numbers of wavelengths are approximately proportional to the index of refraction n, the difference in number of wavelengths is not affected by n at all.

The phase difference between the beams will thus be:

$$\Delta\varphi = 2\pi\Delta N = \frac{8\pi^2 r v}{\lambda c\sqrt{1-\left(\frac{v}{c}\right)^2}} \tag{21}$$

This phase difference is identical to the one in equation (4). So both methods give the same result, as they must.

Ignoring second order terms in v/c, and inserting the area A of the loop and its angular velocity $\omega = v/r$, yields:

$$\Delta\varphi \simeq \frac{8\pi A\,\omega}{\lambda c} \tag{22}$$

which obviously is identical to equation (11).

3 Conclusion

We have shown that the Special Theory of Relativity predicts that the phase difference between the contra-moving light beams in a FOG is:

$$\Delta\varphi = \frac{8\pi A\,\omega}{\lambda c\sqrt{1-\left(\frac{\omega r}{c}\right)^2}}$$

which within any practically possible precision of measurement is consistent with the experimentally verified equation:

$$\Delta\varphi = \frac{8\pi A\,\omega}{\lambda c}$$

The fact that in accordance with experimental evidence, no dependency on the index of refraction in the fibre is predicted, is yet another confirmation of the Special Theory of Relativity.