The twin 'paradox' calculated by the metric

Paul B. Andersen

October 12, 2015

1 Spacetime interval and proper time

The metric equation is generally:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

This is a differential equation. To find the spacetime interval s between two events, one must integrate (1) over a path in spacetime between the events.

When s^2 is positive, the spacetime interval is space-like, when s^2 is negative, the spacetime interval is time-like. The interval between two events on the worldline of an object will always be time-like, in which case we will call the interval *the proper time* of the object.

The differential equation for the proper time can then be written:

$$d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu \tag{2}$$

In flat spacetime the metric tensor is:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

And the equation becomes:

$$(d\tau)^2 = \left(dx^0\right)^2 - \left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2 \tag{4}$$

Alternatively:

$$(c\ d\tau)^2 = (c\ dt)^2 - dx^2 - dy^2 - dz^2 \tag{5}$$

Where τ is the proper time of some object, while $[t, x, y, z]$ are the coordinates of an inertial frame of reference $\mathcal{K}$, and c is the speed of light in vacuum.

If we use the coordinate time t as parameter, the equation can be written:

$$d\tau^2 = \left(1 - \frac{1}{c^2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]\right) dt^2 \tag{6}$$

Which can be simplified to:

$$d\tau = \sqrt{1 - \frac{v(t)^2}{c^2}}\ dt \tag{7}$$

where $\vec{v}(t) \underset{\mathcal{K}}{\rightarrow} \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$ and $v(t)^2 = \vec{v}(t) \cdot \vec{v}(t) = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$.

The proper time between two events on the object's world line will be:

$$\tau = \int_{t_0}^{t_1} \sqrt{1 - \frac{v(t)^2}{c^2}}\ dt \tag{8}$$

Note that $v(t)$ may be any function of t.

2 The twin 'paradox'

2.1 General scenario

Given an inertial frame $\mathcal{K}$ with coordinates $[t, x, y, z]$.
Twin A stays stationary at $x = 0$ in $\mathcal{K}$, while twin B starts from $x = 0$ when A's clock shows 0 and travels to some point $x = L$, where she turns around and travels back to $x = 0$. She is back at the time T as measured in $\mathcal{K}$. Since twin A is stationary in $\mathcal{K}$, her proper time when twin B is back will be $\tau_A = T$.

2.2 B travels with constant speed and instant acceleration

We will assume that twin B travels at the constant speed v and turns abruptly around with a brief, very high acceleration a for a very short time Δt such that $\lim_{\Delta t \to 0} (a\Delta t) = 2v$ and thereafter travels back at the constant speed v.

There are three events of interest. The coordinates of these events in $\mathcal{K}$ are:

$$\begin{array}{rllll} \text{Start } E_0: & t = 0, & x = 0, & y = 0, & z = 0 \\ \text{B turns around } E_1: & t = \frac{T}{2}, & x = \frac{vT}{2}, & y = 0, & z = 0 \\ \text{B is back } E_2: & t = T, & x = 0, & y = 0, & z = 0 \end{array}$$

The proper times of the twins between the events E_0 and E_2 will be:
Twin A:

$$\tau_A = \int_0^T dt = T \tag{9}$$

Twin B:

$$\tau_B = \int_0^{\frac{T}{2}} \sqrt{1 - \frac{v^2}{c^2}}\ dt + \int_{\frac{T}{2}}^{T} \sqrt{1 - \frac{v^2}{c^2}}\ dt = T\sqrt{1 - \frac{v^2}{c^2}} \tag{10}$$

Thus:

$$\tau_B = \tau_A \sqrt{1 - \frac{v^2}{c^2}} \tag{11}$$

2.3 B travels with constant acceleration

Twin B starts from a standstill at the event $E_0 \underset{\mathcal{K}}{\rightarrow} (0,0,0,0)$ and accelerates away with the constant proper acceleration a until the event $E_1 \underset{\mathcal{K}}{\rightarrow} (t_1, x_1, 0, 0)$ when she reverses the direction of the acceleration and accelerates towards the starting point with constant proper acceleration with magnitude a. At the event $E_2 \underset{\mathcal{K}}{\rightarrow} (t_2, L, 0, 0)$ she will be stationary in $\mathcal{K}$, and starts moving towards the starting point. At the event $E_3 \underset{\mathcal{K}}{\rightarrow} (t_3, x_3, 0, 0)$ she reverses the direction of acceleration, and accelerates away from the starting point (brakes). She is stationary at the starting point at the event $E_4 \underset{\mathcal{K}}{\rightarrow} (T, 0, 0, 0)$.

Symmetry makes it obvious that $t_2 = \frac{1}{4}T$, $t_3 = \frac{1}{2}T$, $t_3 = \frac{3}{4}T$ and $x_1 = x_3 = \frac{1}{2}L$.

Let the mass of twin B be m. Let $\mathcal{K}'$ with coordinates $[x', y', z', t']$ be the momentarily co-moving inertial frame of twin B. The x'-axis of $\mathcal{K}'$ is aligned with the x-axis of $\mathcal{K}$, and $\mathcal{K}'$ is moving at the speed v in $\mathcal{K}$.

Since the accelerating force F must be in the same direction as the velocity of $\mathcal{K}'$ in $\mathcal{K}$, the accelerating force must be the same in both frames of reference. The consequence of this is:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}p'}{\mathrm{d}t'} = F \tag{12}$$

Where p is the momentum of twin B in $\mathcal{K}$ and p' is the momentum in $\mathcal{K}'$. We have $p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$ and $\frac{\mathrm{d}p'}{\mathrm{d}t'} = ma$, where v is the speed of twin B along the x-axis of $\mathcal{K}$ and a is the proper acceleration of twin B.

Thus:

$$m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v(t)}{\sqrt{1-\frac{v(t)^2}{c^2}}}\right) = m\,a \tag{13}$$

Twin B is accelerating outwards to $\frac{L}{2}$, $0 \le t \le \frac{T}{4}$

We have $v(0) = 0$:

$$\frac{v(t)}{\sqrt{1-\frac{v(t)^2}{c^2}}} = \int_0^t a\,\mathrm{d}t = at, \tag{14}$$

Solving this equation with respect to v yields:

$$v(t) = \frac{at}{\sqrt{1+\left(\frac{at}{c}\right)^2}} \tag{15}$$

The position of twin B in $\mathcal{K}$ when $x(0) = 0$ is:

$$x(t) = \int_0^t v(t)\,\mathrm{d}t = \frac{c^2}{a}\left(\sqrt{1+\left(\frac{at}{c}\right)^2} - 1\right) \tag{16}$$

$$x\left(\frac{T}{4}\right) = \frac{c^2}{a}\left(\sqrt{1+\left(\frac{aT}{4c}\right)^2}-1\right) = \frac{L}{2} \tag{17}$$

Solving this equation with respect to T yields:

$$T = 2\sqrt{\frac{L^2}{c^2}+\frac{4L}{a}} \tag{18}$$

From equation (7), we have:

$$\frac{\mathrm{d}\tau_B}{\mathrm{d}t} = \sqrt{1-\frac{v(t)^2}{c^2}} = \frac{1}{\sqrt{1+\left(\frac{at}{c}\right)^2}} \tag{19}$$

Since $\tau_B(0) = 0$, we have:

$$\tau_B(t) = \int_0^t \frac{\mathrm{d}t}{\sqrt{1+\left(\frac{at}{c}\right)^2}} = \frac{c}{a}\operatorname{arsinh}\left(\frac{a\,t}{c}\right) \tag{20}$$

$$\tau_B\left(\frac{T}{4}\right) = \frac{c}{a}\operatorname{arsinh}\left(\frac{a\,T}{4\,c}\right) \tag{21}$$

Twin B is braking to L, and then accelerating towards $\frac{L}{2}$, $\quad \frac{T}{4} < t \leq \frac{3T}{4}$

We have $v\left(\frac{T}{2}\right) = 0$:

$$\frac{v(t)}{\sqrt{1-\frac{v(t)^2}{c^2}}} = \int_{\frac{T}{2}}^{t} -a\,\mathrm{d}t = a\left(\frac{T}{2}-t\right) \tag{22}$$

Solving this equation with respect to v yields:

$$v(t) = \frac{a\left(\frac{T}{2}-t\right)}{\sqrt{1+\left(\frac{a\left(\frac{T}{2}-t\right)}{c}\right)^2}} \tag{23}$$

The position of twin B in $\mathcal{K}$ is:

$$x(t) = \int_{\frac{T}{4}}^{t} v(t)\,\mathrm{d}t + x\left(\frac{T}{4}\right) = \frac{c^2}{a}\left(2\sqrt{1+\left(\frac{aT}{4c}\right)^2} - \sqrt{1+\left(\frac{a\left(\frac{T}{2}-t\right)}{c}\right)^2} - 1\right) \tag{24}$$

$$x\left(\frac{T}{2}\right) = \frac{2\,c^2}{a}\left(\sqrt{1+\left(\frac{aT}{4c}\right)^2}-1\right) = L \tag{25}$$

$$x\left(\frac{3\,T}{4}\right) = \frac{c^2}{a}\left(\sqrt{1+\left(\frac{aT}{4c}\right)^2}-1\right) = \frac{L}{2} \tag{26}$$

From equation (7), we have:

$$\frac{\mathrm{d}\tau_B}{\mathrm{d}t} = \sqrt{1 - \frac{v\left(t\right)^2}{c^2}} = \frac{1}{\sqrt{1 + \left(\frac{a\left(\frac{T}{2} - t\right)}{c}\right)^2}} \tag{27}$$

We have:

$$\tau_B\left(t\right) = \int_{\frac{T}{4}}^{t} \frac{\mathrm{d}t}{\sqrt{1 + \left(\frac{a\left(\frac{T}{2} - t\right)}{c}\right)^2}} + \tau_B\left(\frac{T}{4}\right) = \frac{c}{a}\left[\operatorname{arsinh}\left(\frac{a\left(t - \frac{T}{2}\right)}{c}\right) + 2\operatorname{arsinh}\left(\frac{a\,T}{4\,c}\right)\right] \tag{28}$$

$$\tau_B\left(\frac{T}{2}\right) = \frac{2\,c}{a}\operatorname{arsinh}\left(\frac{a\,T}{4\,c}\right) \tag{29}$$

$$\tau_B\left(\frac{3T}{4}\right) = \frac{3\,c}{a}\operatorname{arsinh}\left(\frac{a\,T}{4\,c}\right) \tag{30}$$

Twin B is braking to the starting point, $\frac{3}{4}\mathbf{T} < \mathbf{t} \leq \mathbf{T}$

We have $v\left(T\right) = 0$:

$$\frac{v\left(t\right)}{\sqrt{1 - \frac{v(t)^2}{c^2}}} = \int_{T}^{t} a\,\mathrm{d}t = a\left(t - T\right) \tag{31}$$

Solving this equation with respect to v yields:

$$v\left(t\right) = \frac{a\left(t - T\right)}{\sqrt{1 + \left(\frac{a(t-T)}{c}\right)^2}} \tag{32}$$

The position of twin B in $\mathcal{K}$ is:

$$x\left(t\right) = \int_{\frac{3\,T}{4}}^{t} v\left(t\right)\,\mathrm{d}t + x\left(\frac{3\,T}{4}\right) = \frac{c^2}{a}\left(\sqrt{1 + \left(\frac{a\left(T - t\right)}{c}\right)^2} - 1\right) \tag{33}$$

$$x\left(T\right) = 0 \tag{34}$$

From equation (7), we have:

$$\frac{\mathrm{d}\tau_B}{\mathrm{d}t} = \sqrt{1 - \frac{v\left(t\right)^2}{c^2}} = \frac{1}{\sqrt{1 + \left(\frac{a(t-T)}{c}\right)^2}} \tag{35}$$

We have:

$$\tau_B\left(t\right) = \int_{\frac{3T}{4}}^{t} \frac{\mathrm{d}t}{\sqrt{1 + \left(\frac{a(t-T)}{c}\right)^2}} + \tau_B\left(\frac{3\,T}{4}\right) = \frac{c}{a}\left[\operatorname{arsinh}\left(\frac{a\,\left(t - T\right)}{c}\right) + 4\operatorname{arsinh}\left(\frac{a\,T}{4\,c}\right)\right] \tag{36}$$

$$\tau_B\left(T\right) = \frac{4\,c}{a}\operatorname{arsinh}\left(\frac{a\,T}{4\,c}\right) \tag{37}$$

Summing up

The speed:

$$v(t) = \frac{at}{\sqrt{1+\left(\frac{at}{c}\right)^2}} \qquad 0 \le t \le \frac{T}{4} \tag{38}$$

$$v(t) = \frac{a\left(\frac{T}{2}-t\right)}{\sqrt{1+\left(\frac{a\left(\frac{T}{2}-t\right)}{c}\right)^2}} \qquad \frac{T}{4} < t \le \frac{3T}{4} \tag{39}$$

$$v(t) = \frac{a(t-T)}{\sqrt{1+\left(\frac{a(t-T)}{c}\right)^2}} \qquad \frac{3}{4}T < t \le T \tag{40}$$

The position:

$$x(t) = \frac{c^2}{a}\left(\sqrt{1+\left(\frac{at}{c}\right)^2}-1\right) \qquad 0 \le t \le \frac{T}{4} \tag{41}$$

$$x(t) = \frac{c^2}{a}\left(2\sqrt{1+\left(\frac{aT}{4c}\right)^2}-\sqrt{1+\left(\frac{a\left(\frac{T}{2}-t\right)}{c}\right)^2}-1\right) \qquad \frac{T}{4} < t \le \frac{3T}{4} \tag{42}$$

$$x(t) = \frac{c^2}{a}\left(\sqrt{1+\left(\frac{a(T-t)}{c}\right)^2}-1\right) \qquad \frac{3}{4}T < t \le T \tag{43}$$

The rate of twin B's clock as observed by twin A:

$$\frac{\mathrm{d}\tau_B}{\mathrm{d}t} = \frac{1}{\sqrt{1+\left(\frac{at}{c}\right)^2}} \qquad 0 \le t \le \frac{T}{4} \tag{44}$$

$$\frac{\mathrm{d}\tau_B}{\mathrm{d}t} = \frac{1}{\sqrt{1+\left(\frac{a\left(\frac{T}{2}-t\right)}{c}\right)^2}} \qquad \frac{T}{4} < t \le \frac{3T}{4} \tag{45}$$

$$\frac{\mathrm{d}\tau_B}{\mathrm{d}t} = \frac{1}{\sqrt{1+\left(\frac{a(t-T)}{c}\right)^2}} \qquad \frac{3}{4}T < t \le T \tag{46}$$

The proper time of twin B's clock :

$$\tau_B(t) = \frac{c}{a}\operatorname{arsinh}\left(\frac{a\,t}{c}\right) \qquad 0 \le t \le \frac{T}{4} \tag{47}$$

$$\tau_B(t) = \frac{c}{a}\left[\operatorname{arsinh}\left(\frac{a\left(t-\frac{T}{2}\right)}{c}\right)+2\operatorname{arsinh}\left(\frac{a\,T}{4\,c}\right)\right] \qquad \frac{T}{4} < t \le \frac{3T}{4} \tag{48}$$

$$\tau_B(t) = \frac{c}{a}\left[\operatorname{arsinh}\left(\frac{a\,(t-T)}{c}\right)+4\operatorname{arsinh}\left(\frac{a\,T}{4\,c}\right)\right] \qquad \frac{3}{4}T < t \le T \tag{49}$$

2.4 Concrete example

We will use the following units:

distance:	light year	$[ly]$
time:	year	$[y]$
speed:	light year per year	$[\frac{ly}{y}]$
acceleration:	speed per year	$[\frac{ly}{y^2}]$

We will calculate the twin 'paradox' scenario in chapter 2.2 with $L = 10\,ly$ and $a = c$ per year $= 1\frac{ly}{y^2}$.

Equation (18) gives:

$$T = 2\sqrt{\frac{L^2}{c^2} + \frac{4L}{a}} = 23.664\,y$$

Using these numbers in equations (38),(39) and (40) and gives the following speed of twin B as a function of the time:

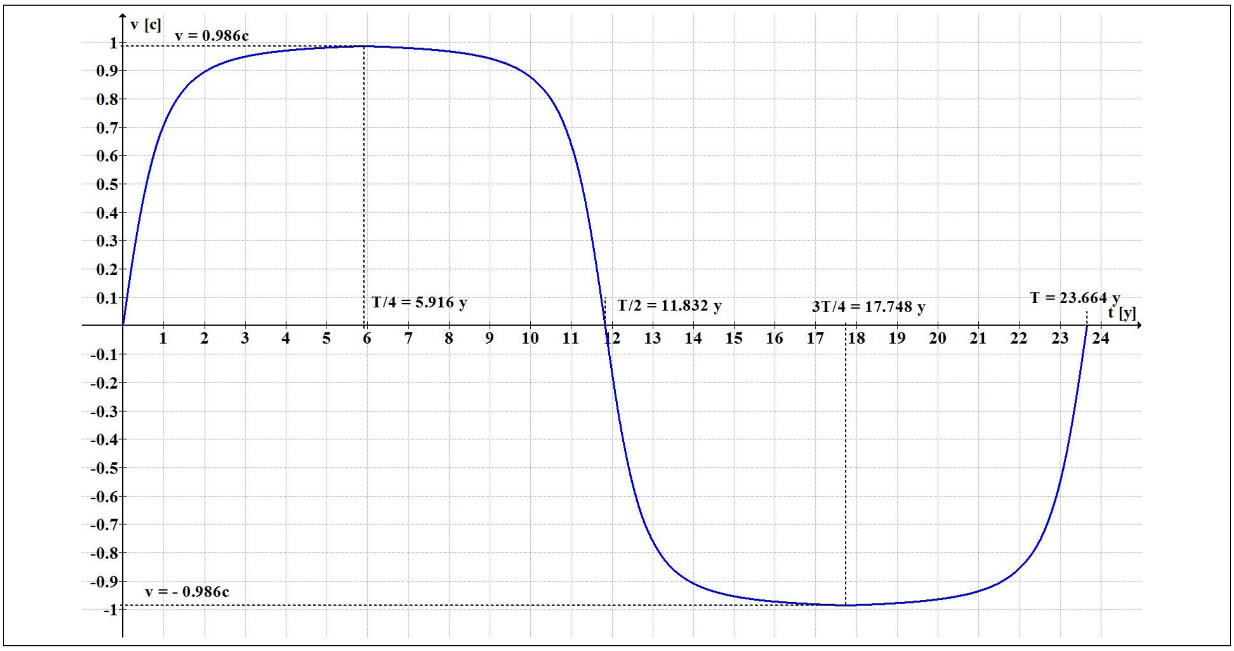

Figure 1: *The speed of twin B in $\mathcal{K}$*

Equations (41),(42) and (43) give the following position of twin B as a function of the time:

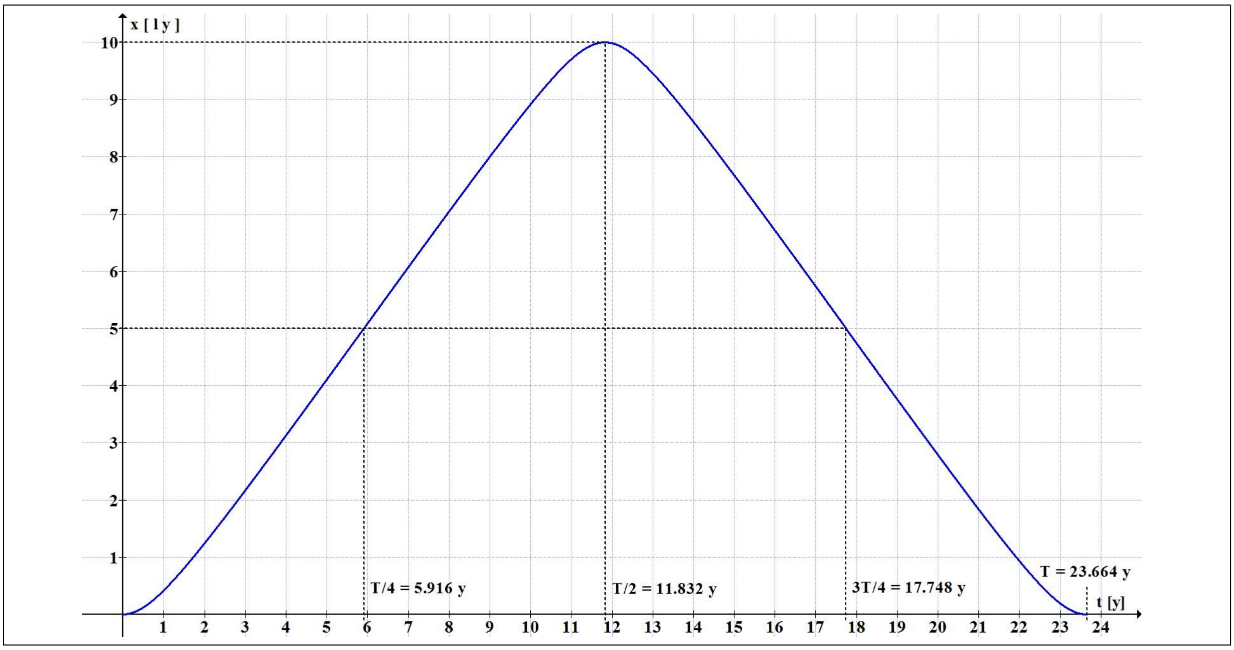

Figure 2: *The position of twin B in $\mathcal{K}$*

Equations (44),(45) and (46) give the following rate of twin B's clock as observed by twin A, as a function of the time:

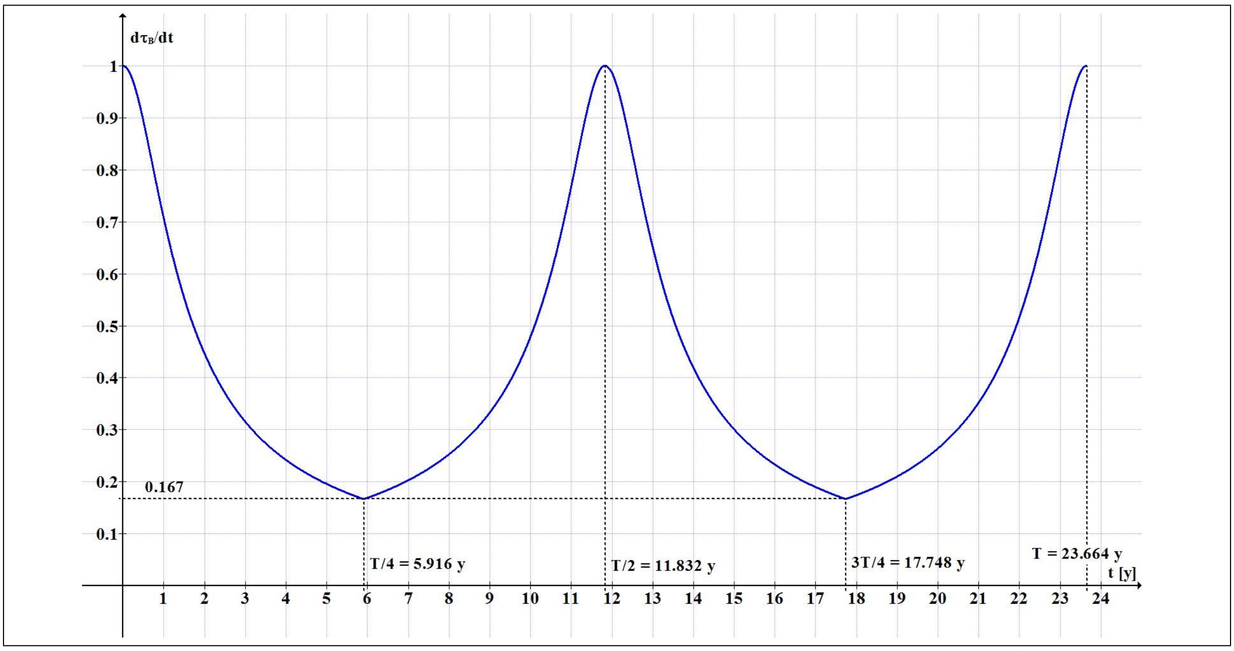

Figure 3: *The rate of twin B's clock as observed by twin A.*

Equations (47),(48) and (49) give the proper time of twin B as a function of the time in $\mathcal{K}$.

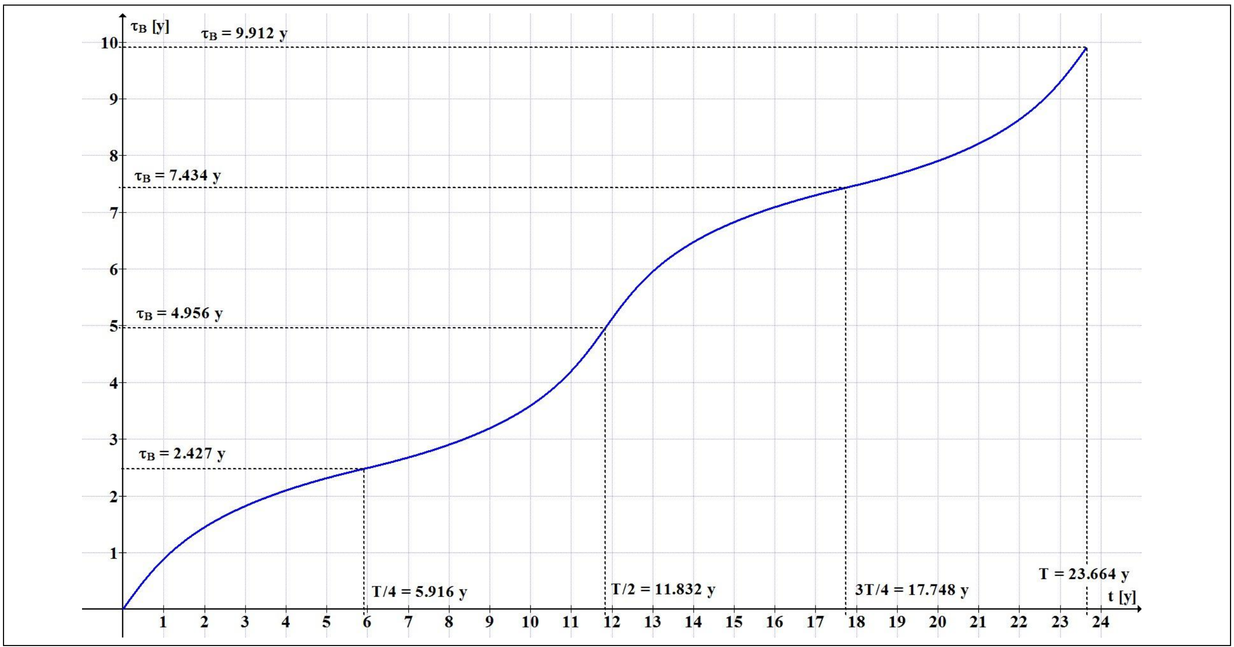

Figure 4: *The proper time of twin B as a function of the time in $\mathcal{K}$*

3 A twin 'paradox' simulation

On my homepage https://paulba.no/ you will find a twin paradox simulation:

Run the twin 'paradox' simulation

The figure below is a picture of the screen when this simulation is run with the same parameters as in chapter 2.4.

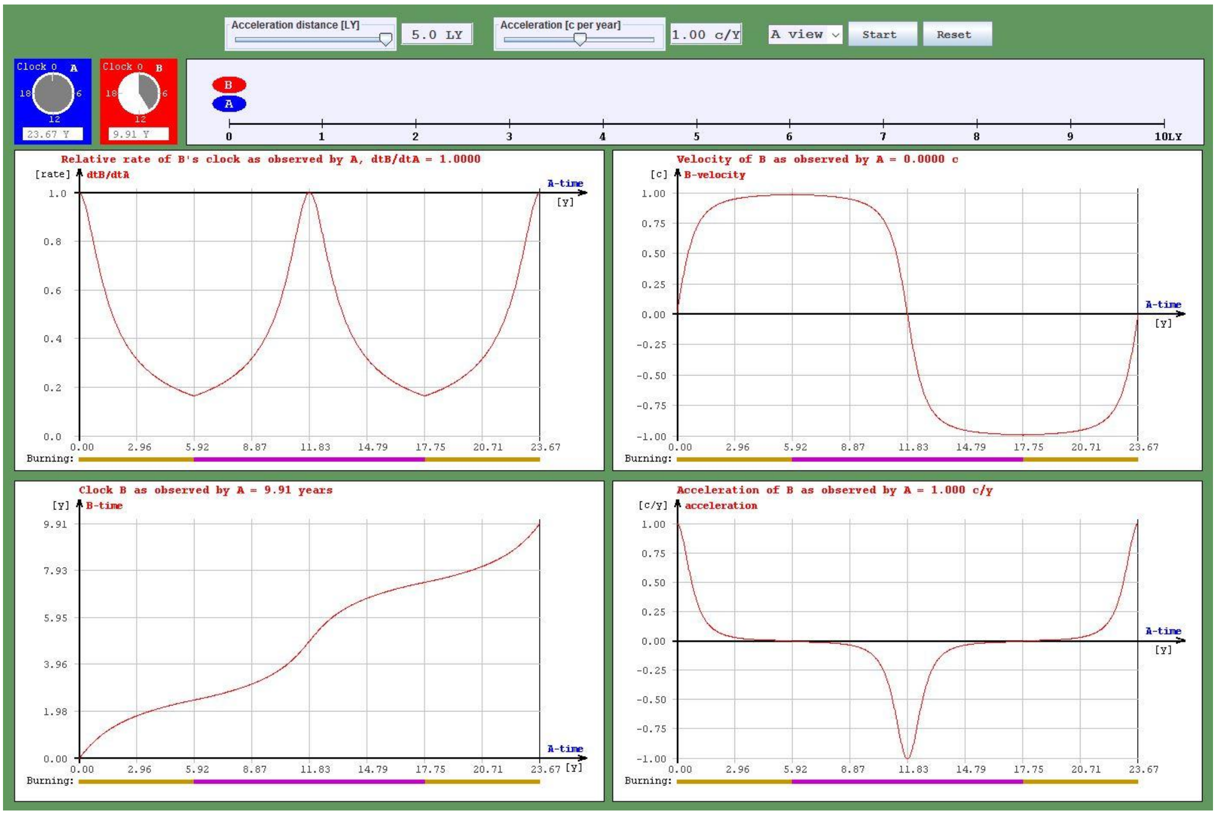

Figure 5: *A run of the twin paradox simulation with parameters as in chapter 2.4*

Compare this to the figures in chapter 2.4.