The twin 'paradox' calculated by the metric

Paul B. Andersen

October 12, 2015

1 Spacetime interval and proper time

The metric equation is generally:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{1}$$

This is a differential equation. To find the spacetime interval s between two events, one must integrate (1) over a path in spacetime between the events.

When s^2 is positive, the spacetime interval is space-like, when s^2 is negative, the spacetime interval is time-like. The interval between two events on the worldline of an object will always be time-like, in which case we will call the interval *the proper time* of the object.

The differential equation for the proper time can then be written:

$$\mathrm{d}\tau^2 = -g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{2}$$

In flat spacetime the metric tensor is:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
(3)

And the equation becomes:

$$(d\tau)^{2} = (dx^{0})^{2} - (dx^{1})^{2} - (dx^{2})^{2} - (dx^{3})^{2}$$
(4)

Alternatively:

$$(c \cdot \mathrm{d}\tau)^2 = (c \cdot \mathrm{d}t)^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \tag{5}$$

Where τ is the proper time of some object, while [t, x, y, z] are the coordinates of an inertial frame of reference \mathcal{K} , and c is the speed of light in vacuum.

If we use the coordinate time t as parameter, the equation can be written:

$$d\tau^{2} = \left(1 - \frac{1}{c^{2}} \left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{2} + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^{2} + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^{2} \right] \right) \mathrm{d}t^{2}$$
(6)

Which can be simplified to:

$$d\tau = \sqrt{1 - \frac{v(t)^2}{c^2}} dt$$
(7)

where $\vec{v}(t) \xrightarrow{\mathcal{K}} \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right)$ and $v(t)^2 = \vec{v}(t) \cdot \vec{v}(t) = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2$.

The proper time between two events on the object's world line will be:

$$\tau = \int_{t_0}^{t_1} \sqrt{1 - \frac{v(t)^2}{c^2}} \, \mathrm{d}t \tag{8}$$

Note that v(t) may be any function of t.

2 The twin 'paradox'

2.1 General scenario

Given an inertial frame \mathcal{K} with coordinates [t, x, y, z].

Twin A stays stationary at x = 0 in \mathcal{K} , while twin B starts from x = 0 when A's clock shows 0 and travels to some point x = L, where she turns around and travels back to x = 0. She is back at the time T as measured in \mathcal{K} . Since twin A is stationary in \mathcal{K} , her proper time when twin B is back will be $\tau_A = T$.

2.2 B travels with constant speed and instant acceleration

We will assume that twin B travels at the constant speed v and turns abruptly around with a brief, very high acceleration a for a very short time Δt such that $\lim_{\Delta t \to 0} (a\Delta t) = 2v$ and thereafter travels back at the constant speed v.

There are three events of interest. The coordinates of these events in \mathcal{K} are:

Start
$$E_0:$$
 $t = 0,$ $x = 0,$ $y = 0,$ $z = 0$
B turns around $E_1:$ $t = \frac{T}{2},$ $x = \frac{vT}{2},$ $y = 0,$ $z = 0$
B is back $E_2:$ $t = T,$ $x = 0,$ $y = 0,$ $z = 0$

The proper times of the twins between the events E_0 and E_2 will be: Twin A:

$$\tau_A = \int_0^T \mathrm{d}t = T \tag{9}$$

Twin B:

$$\tau_B = \int_0^{\frac{T}{2}} \sqrt{1 - \frac{v^2}{c^2}} \, \mathrm{d}t + \int_{\frac{T}{2}}^T \sqrt{1 - \frac{v^2}{c^2}} \, \mathrm{d}t = T \sqrt{1 - \frac{v^2}{c^2}} \tag{10}$$

Thus:

$$\tau_B = \tau_A \sqrt{1 - \frac{v^2}{c^2}} \tag{11}$$

2.3 B travels with constant acceleration

Twin *B* starts from a standstill at the event $E_0 \xrightarrow{\kappa} (0, 0, 0, 0)$ and accelerates away with the constant proper acceleration *a* until the event $E_1 \xrightarrow{\kappa} (t_1, x_1, 0, 0)$ when she reverses the direction of the acceleration and accelerates towards the starting point with constant proper acceleration with magnitude *a*. At the event $E_2 \xrightarrow{\kappa} (t_2, L, 0, 0)$ she will be stationary in \mathcal{K} , and starts moving towards the starting point. At the event $E_3 \xrightarrow{\kappa} (t_3, x_3, 0, 0)$ she reverses the direction of acceleration, and accelerates away from the starting point (brakes). She is stationary at the starting point at the event $E_4 \xrightarrow{\kappa} (T, 0, 0, 0)$.

Symmetry makes it obvious that $t_1 = \frac{1}{4}T$, $t_2 = \frac{1}{2}T$, $t_3 = \frac{3}{4}T$ and $x_1 = x_3 = \frac{1}{2}L$.

Let the mass of twin B be m. Let \mathcal{K}' with coordinates [x', y', z', t'] be the momentarily co-moving inertial frame of twin B. The x'-axis of \mathcal{K}' is aligned with the x-axis of \mathcal{K} , and \mathcal{K}' is moving at the speed v in \mathcal{K} .

Since the accelerating force F must be in the same direction as the velocity of \mathcal{K}' in \mathcal{K} , the accelerating force must be the same in both frames of reference. The consequence of this is:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}p'}{\mathrm{d}t'} = F \tag{12}$$

Where p is the momentum of twin B in \mathcal{K} and p' is the momentum in \mathcal{K}' . We have $p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$ and $\frac{dp'}{dt'} = ma$, where v is the speed of twin B along the x-axis of \mathcal{K} and a is the proper conclusion of twin B

proper acceleration of twin B.

Thus:

$$m \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{v\left(t\right)}{\sqrt{1 - \frac{v\left(t\right)^{2}}{c^{2}}}} \right) = m a \tag{13}$$

Twin B is accelerating outwards to $\frac{L}{2}$, $0 \le t \le \frac{T}{4}$

We have v(0) = 0:

$$\frac{v(t)}{\sqrt{1 - \frac{v(t)^2}{c^2}}} = \int_0^t a \, \mathrm{d}t = at,\tag{14}$$

Solving this equation with respect to v yields:

$$v\left(t\right) = \frac{at}{\sqrt{1 + \left(\frac{at}{c}\right)^2}}\tag{15}$$

The position x(t) of twin B in \mathcal{K} when x(0) = 0 is:

$$x(t) = \int_0^t v(t) \, \mathrm{d}t = \frac{c^2}{a} \left(\sqrt{1 + \left(\frac{at}{c}\right)^2} - 1 \right)$$
(16)

$$x\left(\frac{T}{4}\right) = \frac{c^2}{a}\left(\sqrt{1 + \left(\frac{aT}{4c}\right)^2} - 1\right) = \frac{L}{2}$$
(17)

Solving this equation with respect to T yields:

$$T = 2\sqrt{\frac{L^2}{c^2} + \frac{4L}{a}} \tag{18}$$

From equation (7), we have:

$$\frac{\mathrm{d}\tau_B}{\mathrm{d}t} = \sqrt{1 - \frac{v\left(t\right)^2}{c^2}} = \frac{1}{\sqrt{1 + \left(\frac{at}{c}\right)^2}} \tag{19}$$

Since $\tau_B(0) = 0$, we have:

$$\tau_B(t) = \int_0^t \frac{\mathrm{d}t}{\sqrt{1 + \left(\frac{at}{c}\right)^2}} = \frac{c}{a} \operatorname{arsinh}\left(\frac{at}{c}\right)$$
(20)

$$\tau_B\left(\frac{T}{4}\right) = \frac{c}{a}\operatorname{arsinh}\left(\frac{aT}{4c}\right) \tag{21}$$

Twin B is braking to L, and then accelerating towards $\frac{L}{2}$, $\frac{T}{4} < t \le \frac{3T}{4}$ We have $v\left(\frac{T}{2}\right) = 0$:

$$\frac{v(t)}{\sqrt{1 - \frac{v(t)^2}{c^2}}} = \int_{\frac{T}{2}}^t -a \, \mathrm{d}t = a\left(\frac{T}{2} - t\right) \tag{22}$$

Solving this equation with respect to v yields:

$$v(t) = \frac{a\left(\frac{T}{2} - t\right)}{\sqrt{1 + \left(\frac{a\left(\frac{T}{2} - t\right)}{c}\right)^2}}$$
(23)

The position of twin B in \mathcal{K} is:

$$x(t) = \int_{\frac{T}{4}}^{t} v(t) \, \mathrm{d}t + x\left(\frac{T}{4}\right) = \frac{c^2}{a} \left(2\sqrt{1 + \left(\frac{aT}{4c}\right)^2} - \sqrt{1 + \left(\frac{a\left(\frac{T}{2} - t\right)}{c}\right)^2} - 1\right)$$
(24)

$$x\left(\frac{T}{2}\right) = \frac{2c^2}{a} \left(\sqrt{1 + \left(\frac{aT}{4c}\right)^2} - 1\right) = L$$
(25)

$$x\left(\frac{3T}{4}\right) = \frac{c^2}{a}\left(\sqrt{1+\left(\frac{aT}{4c}\right)^2} - 1\right) = \frac{L}{2}$$
(26)

From equation (7), we have:

$$\frac{\mathrm{d}\tau_B}{\mathrm{d}t} = \sqrt{1 - \frac{v\left(t\right)^2}{c^2}} = \frac{1}{\sqrt{1 + \left(\frac{a\left(\frac{T}{2} - t\right)}{c}\right)^2}}$$
(27)

We have:

$$\tau_B(t) = \int_{\frac{T}{4}}^{t} \frac{\mathrm{d}t}{\sqrt{1 + \left(\frac{a(\frac{T}{2} - t)}{c}\right)^2}} + \tau_B\left(\frac{T}{4}\right) = \frac{c}{a} \left[\operatorname{arsinh}\left(\frac{a\left(t - \frac{T}{2}\right)}{c}\right) + 2\operatorname{arsinh}\left(\frac{aT}{4c}\right) \right]$$
(28)

$$\tau_B\left(\frac{T}{2}\right) = \frac{2c}{a}\operatorname{arsinh}\left(\frac{aT}{4c}\right) \tag{29}$$

$$\tau_B\left(\frac{3T}{4}\right) = \frac{3c}{a}\operatorname{arsinh}\left(\frac{aT}{4c}\right) \tag{30}$$

Twin B is braking to the starting point, $\frac{3}{4}T < t \le T$ We have v(T) = 0:

$$\frac{v(t)}{\sqrt{1 - \frac{v(t)^2}{c^2}}} = \int_T^t a \, \mathrm{d}t = a \, (t - T) \tag{31}$$

Solving this equation with respect to v yields:

$$v(t) = \frac{a(t-T)}{\sqrt{1 + \left(\frac{a(t-T)}{c}\right)^2}}$$
(32)

The position of twin B in \mathcal{K} is:

$$x(t) = \int_{\frac{3T}{4}}^{t} v(t) \, \mathrm{d}t + x\left(\frac{3T}{4}\right) = \frac{c^2}{a} \left(\sqrt{1 + \left(\frac{a(T-t)}{c}\right)^2} - 1\right)$$
(33)

$$x\left(T\right) = 0\tag{34}$$

From equation (7), we have:

$$\frac{\mathrm{d}\tau_B}{\mathrm{d}t} = \sqrt{1 - \frac{v\left(t\right)^2}{c^2}} = \frac{1}{\sqrt{1 + \left(\frac{a\left(t-T\right)}{c}\right)^2}}$$
(35)

We have:

$$\tau_B(t) = \int_{\frac{3T}{4}}^{t} \frac{\mathrm{d}t}{\sqrt{1 + \left(\frac{a(t-T)}{c}\right)^2}} + \tau_B\left(\frac{3T}{4}\right) = \frac{c}{a} \left[\operatorname{arsinh}\left(\frac{a(t-T)}{c}\right) + 4\operatorname{arsinh}\left(\frac{aT}{4c}\right) \right] \quad (36)$$

$$\tau_B(T) = \frac{4c}{a} \operatorname{arsinh}\left(\frac{aT}{4c}\right) \tag{37}$$

Summing up

The speed:

$$v(t) = \frac{at}{\sqrt{1 + \left(\frac{at}{c}\right)^2}} \qquad \qquad 0 \le t \le \frac{T}{4}$$
(38)

$$v(t) = \frac{a(\frac{1}{2} - t)}{\sqrt{1 + \left(\frac{a(\frac{T}{2} - t)}{c}\right)^2}} \qquad \qquad \frac{T}{4} < t \le \frac{3T}{4}$$
(39)

$$v(t) = \frac{a(t-T)}{\sqrt{1 + \left(\frac{a(t-T)}{c}\right)^2}} \qquad \qquad \frac{3}{4}T < t \le T$$

$$\tag{40}$$

The position:

$$x(t) = \frac{c^2}{a} \left(\sqrt{1 + \left(\frac{at}{c}\right)^2} - 1 \right) \qquad \qquad 0 \le t \le \frac{T}{4}$$
(41)

$$x(t) = \frac{c^2}{a} \left(2\sqrt{1 + \left(\frac{aT}{4c}\right)^2} - \sqrt{1 + \left(\frac{a\left(\frac{T}{2} - t\right)}{c}\right)^2} - 1 \right) \qquad \qquad \frac{T}{4} < t \le \frac{3T}{4} \qquad (42)$$

$$x(t) = \frac{c^2}{a} \left(\sqrt{1 + \left(\frac{a(T-t)}{c}\right)^2} - 1 \right) \qquad \frac{3}{4}T < t \le T \qquad (43)$$

The rate of twin B's clock as observed by twin A:

$$\frac{\mathrm{d}\tau_B}{\mathrm{d}t} = \frac{1}{\sqrt{1 + \left(\frac{at}{c}\right)^2}} \qquad \qquad 0 \le t \le \frac{T}{4} \tag{44}$$

$$\frac{\mathrm{d}\tau_B}{\mathrm{d}t} = \frac{1}{\sqrt{1 + \left(\frac{a(\frac{T}{2} - t)}{c}\right)^2}} \qquad \qquad \frac{T}{4} < t \le \frac{3T}{4} \tag{45}$$

$$\frac{\mathrm{d}\tau_B}{\mathrm{d}t} = \frac{1}{\sqrt{1 + \left(\frac{a(t-T)}{c}\right)^2}} \qquad \qquad \frac{3}{4}T < t \le T$$
(46)

The proper time of twin B's clock :

$$\tau_B(t) = \frac{c}{a} \operatorname{arsinh}\left(\frac{a\,t}{c}\right) \qquad \qquad 0 \le t \le \frac{T}{4} \qquad (47)$$

$$\tau_B(t) = \frac{c}{a} \left[\operatorname{arsinh}\left(\frac{a\left(t - \frac{T}{2}\right)}{c}\right) + 2\operatorname{arsinh}\left(\frac{aT}{4c}\right) \right] \qquad \qquad \frac{T}{4} < t \le \frac{3T}{4} \qquad (48)$$

$$\tau_B(t) = \frac{c}{a} \left[\operatorname{arsinh}\left(\frac{a(t-T)}{c}\right) + 4\operatorname{arsinh}\left(\frac{aT}{4c}\right) \right] \qquad \frac{3}{4}T < t \le T$$
(49)

2.4 Concrete example

We will use the following units:

distance:	light year	[ly]
time:	year	[y]
speed:	light year per year	$\left[\frac{ly}{y}\right]$
acceleration:	speed per year	$\left[\frac{ly}{u^2}\right]$

We will calculate the twin 'paradox' scenario in chapter 2.2 with $L=10\,ly$ and a=c per year $=1\frac{ly}{y^2}.$

Equation (18) gives:

$$T = 2\sqrt{\frac{L^2}{c^2} + \frac{4L}{a}} = 23.664\,y$$

Using these numbers in equations (38),(39) and (40) and gives the following speed of twin B as a function of the time:



Figure 1: The speed of twin B in \mathcal{K}



Equations (41),(42) and (43) give the following position of twin B as a function of the time:

i igure 2. The position of tant D in te

Equations (44),(45) and (46) give the following rate of twin B's clock as observed by twin A, as a function of the time:





Equations (47),(48) and (49) give the proper time of twin B as a function of the time in \mathcal{K} .

Figure 4: The proper time of twin B as a function of the time in \mathcal{K}

3 A twin 'paradox' simulation

On my homepage https://paulba.no/ you will find a twin paradox simulation:

Run the twin 'paradox' simulation

The figure below is a picture of the screen when this simulation is run with the same parameters as in chapter 2.4.



Figure 5: A run of the twin paradox simulation with parameters as in chapter 2.4 Compare this to the figures in chapter 2.4.