The twin 'paradox' calculated by the metric

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1 Spacetime interval and proper time

The metric equation is generally:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

(1)

This is a differential equation. To find the spacetime interval $s$ between two events, one must integrate (1) over a path in spacetime between the events.

When $s^2$ is positive, the spacetime interval is space-like, when $s^2$ is negative, the spacetime interval is time-like. The interval between two events on the worldline of an object will always be time-like, in which case we will call the interval the proper time of the object.

The differential equation for the proper time can then be written:

$$d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$$

(2)

In flat spacetime the metric tensor is:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(3)

And the equation becomes:

$$(d\tau)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

(4)

Alternatively:

$$(c \, d\tau)^2 = (c \, dt)^2 - dx^2 - dy^2 - dz^2$$

(5)

Where $\tau$ is the proper time of some object, while $[t, x, y, z]$ are the coordinates of an inertial frame of reference $\mathcal{K}$, and $c$ is the speed of light in vacuum.

If we use the coordinate time $t$ as parameter, the equation can be written:

$$d\tau^2 = \left(1 - \frac{1}{c^2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]\right)dt^2$$

(6)
Which can be simplified to:

\[
d\tau = \sqrt{1 - \frac{v(t)^2}{c^2}} \, dt
\]  

(7)

where \( \vec{v}(t) = \left( \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) \) and \( v(t)^2 = \vec{v}(t) \cdot \vec{v}(t) = \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \).

The proper time between two events on the object’s world line will be:

\[
\tau = \int_{t_0}^{t_1} \sqrt{1 - \frac{v(t)^2}{c^2}} \, dt
\]  

(8)

Note that \( v(t) \) may be any function of \( t \).

2 The twin ’paradox’

2.1 General scenario

Given an inertial frame \( \mathcal{K} \) with coordinates \([t, x, y, z]\).

Twin \( A \) stays stationary at \( x = 0 \) in \( \mathcal{K} \), while twin \( B \) starts from \( x = 0 \) when \( A \)’s clock shows 0 and travels to some point \( x = L \), where she turns around and travels back to \( x = 0 \). She is back at the time \( T \) as measured in \( \mathcal{K} \). Since twin \( A \) is stationary in \( \mathcal{K} \), her proper time when twin \( B \) is back will be \( \tau_A = T \).

2.2 B travels with constant speed and instant acceleration

We will assume that twin \( B \) travels at the constant speed \( v \) and turns abruptly around with a brief, very high acceleration \( a \) for a very short time \( \Delta t \) such that \( \lim_{\Delta t \to 0} (a\Delta t) = 2v \) and thereafter travels back at the constant speed \( v \).

There are three events of interest. The coordinates of these events in \( \mathcal{K} \) are:

- Start \( E_0 \): \( t = 0, \quad x = 0, \quad y = 0, \quad z = 0 \)
- B turns around \( E_1 \): \( t = \frac{T}{2}, \quad x = \frac{vT}{2}, \quad y = 0, \quad z = 0 \)
- B is back \( E_2 \): \( t = T, \quad x = 0, \quad y = 0, \quad z = 0 \)

The proper times of the twins between the events \( E_0 \) and \( E_2 \) will be:

Twin A:

\[
\tau_A = \int_0^T dt = T
\]  

(9)

Twin B:

\[
\tau_B = \int_0^{T_2} \sqrt{1 - \frac{v^2}{c^2}} \, dt + \int_{T_2}^T \sqrt{1 - \frac{v^2}{c^2}} \, dt = T \sqrt{1 - \frac{v^2}{c^2}}
\]  

(10)

Thus:

\[
\tau_B = \tau_A \sqrt{1 - \frac{v^2}{c^2}}
\]  

(11)
2.3 B travels with constant acceleration

Twin B starts from a standstill at the event \( E_0 \rightarrow (0,0,0,0) \) and accelerates away with the constant proper acceleration \( a \) until the event \( E_1 \rightarrow (t_1,x_1,0,0) \) when she reverses the direction of the acceleration and accelerates towards the starting point with constant proper acceleration with magnitude \( a \). At the event \( E_2 \rightarrow (t_2,L,0,0) \) she will be stationary in \( \mathcal{K} \), and starts moving towards the starting point. At the event \( E_3 \rightarrow (t_3,x_3,0,0) \) she reverses the direction of acceleration, and accelerates away from the starting point (brakes). She is stationary at the starting point at the event \( E_4 \rightarrow (T,0,0,0) \).

Symmetry makes it obvious that \( t_2 = \frac{1}{4}T \), \( t_3 = \frac{1}{2}T \), \( t_3 = \frac{3}{4}T \) and \( x_1 = x_3 = \frac{1}{2}L \).

Let the mass of twin B be \( m \). Let \( \mathcal{K}' \) with coordinates \([x',y',z',t']\) be the momentarily co-moving inertial frame of twin B. The \( x' \)-axis of \( \mathcal{K}' \) is aligned with the \( x \)-axis of \( \mathcal{K} \), and \( \mathcal{K}' \) is moving at the speed \( v \) in \( \mathcal{K} \).

Since the accelerating force \( F \) must be in the same direction as the velocity of \( \mathcal{K}' \) in \( \mathcal{K} \), the accelerating force must be the same in both frames of reference. The consequence of this is:

\[
\frac{dp}{dt} = \frac{dp'}{dt'} = F
\]  

(12)

Where \( p \) is the momentum of twin B in \( \mathcal{K} \) and \( p' \) is the momentum in \( \mathcal{K}' \). We have \( p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \) and \( \frac{dp'}{dt'} = ma \), where \( v \) is the speed of twin B along the \( x \)-axis of \( \mathcal{K} \) and \( a \) is the proper acceleration of twin B.

Thus:

\[
m \frac{d}{dt} \left( \frac{v(t)}{\sqrt{1 - \frac{v(t)^2}{c^2}}} \right) = ma
\]  

(13)

Twin B is accelerating outwards to \( \frac{L}{2} \), \( 0 \leq t \leq \frac{T}{4} \)

We have \( v(0) = 0 \):

\[
\frac{v(t)}{\sqrt{1 - \frac{v(t)^2}{c^2}}} = \int_0^t a \, dt = at,
\]  

(14)

Solving this equation with respect to \( v \) yields:

\[
v(t) = \frac{at}{\sqrt{1 + \left( \frac{at}{c} \right)^2}}
\]  

(15)

The position of twin B in \( \mathcal{K} \) when \( x(0) = 0 \) is:

\[
x(t) = \int_0^t v(t) \, dt = \frac{c^2}{a} \left( \sqrt{1 + \left( \frac{at}{c} \right)^2} - 1 \right)
\]  

(16)
\[ x\left(\frac{T}{4}\right) = \frac{c^2}{a} \left( \sqrt{1 + \left(\frac{aT}{4c}\right)^2} - 1 \right) = \frac{L}{2} \quad (17) \]

Solving this equation with respect to \( T \) yields:
\[ T = 2 \sqrt{\frac{L^2}{c^2} + \frac{4L}{a}} \quad (18) \]

From equation (7), we have:
\[ \frac{d\tau_B}{dt} = \sqrt{1 - \frac{v(t)^2}{c^2}} = \frac{1}{\sqrt{1 + \left(\frac{at}{c}\right)^2}} \quad (19) \]

Since \( \tau_B(0) = 0 \), we have:
\[ \tau_B(t) = \int_0^t \frac{dt}{\sqrt{1 + \left(\frac{at}{c}\right)^2}} = \frac{c}{a} \text{arsinh}\left(\frac{at}{c}\right) \quad (20) \]
\[ \tau_B\left(\frac{T}{4}\right) = \frac{c}{a} \text{arsinh}\left(\frac{aT}{4c}\right) \quad (21) \]

Twin B is braking to L, and then accelerating towards \( \frac{T}{2} \), \( \frac{T}{4} < t \leq \frac{3T}{4} \)

We have \( v\left(\frac{T}{2}\right) = 0 \):
\[ \frac{v(t)}{\sqrt{1 - \frac{v(t)^2}{c^2}}} = \int_{\frac{T}{2}}^t -a \, dt = a\left(\frac{T}{2} - t\right) \quad (22) \]

Solving this equation with respect to \( v \) yields:
\[ v(t) = \frac{a\left(\frac{T}{2} - t\right)}{\sqrt{1 + \left(\frac{a\left(\frac{T}{2} - t\right)}{c}\right)^2}} \quad (23) \]

The position of twin B in \( \mathcal{K} \) is:
\[ x(t) = \int_{\frac{T}{4}}^t v(t) \, dt + x\left(\frac{T}{4}\right) = \frac{c^2}{a} \left( 2\sqrt{1 + \left(\frac{aT}{4c}\right)^2} - \sqrt{1 + \left(\frac{a\left(\frac{T}{2} - t\right)}{c}\right)^2} - 1 \right) \quad (24) \]
\[ x\left(\frac{T}{2}\right) = \frac{2c^2}{a} \left( \sqrt{1 + \left(\frac{aT}{4c}\right)^2} - 1 \right) = L \quad (25) \]
\[ x\left(\frac{3T}{4}\right) = \frac{c^2}{a} \left( \sqrt{1 + \left(\frac{aT}{4c}\right)^2} - 1 \right) = \frac{L}{2} \quad (26) \]
From equation (7), we have:

\[
\frac{d\tau_B}{dt} = \sqrt{1 - \frac{v(t)^2}{c^2}} = \frac{1}{\sqrt{1 + \left(\frac{a(t-T)}{c}\right)^2}}
\]  

(27)

We have:

\[
\tau_B(t) = \int_{\frac{T}{4}}^{t} \frac{dt}{\sqrt{1 + \left(\frac{a(t-T)}{c}\right)^2}} + \tau_B\left(\frac{T}{4}\right) = \frac{c}{a} \left[ \text{arsinh}\left(\frac{a(t - \frac{T}{2})}{c}\right) + 2\text{arsinh}\left(\frac{aT}{4c}\right) \right]
\]

(28)

\[
\tau_B\left(\frac{T}{2}\right) = \frac{2c}{a} \text{arsinh}\left(\frac{aT}{4c}\right)
\]

(29)

\[
\tau_B\left(\frac{3T}{4}\right) = \frac{3c}{a} \text{arsinh}\left(\frac{aT}{4c}\right)
\]

(30)

**Twin B is braking to the starting point, \( \frac{3}{4} T < t \leq T \)**

We have \( v(T) = 0 \):

\[
\frac{v(t)}{\sqrt{1 - \frac{v(t)^2}{c^2}}} = \int_{T}^{t} a \, dt = a(t - T)
\]

(31)

Solving this equation with respect to \( v \) yields:

\[
v(t) = \frac{a(t - T)}{\sqrt{1 + \left(\frac{a(t-T)}{c}\right)^2}}
\]

(32)

The position of twin B in \( \mathcal{K} \) is:

\[
x(t) = \int_{\frac{3T}{4}}^{t} v(t) \, dt + x\left(\frac{3T}{4}\right) = \frac{c^2}{a} \left( \sqrt{1 + \left(\frac{a(T-t)}{c}\right)^2} - 1 \right)
\]

(33)

\[x(T) = 0\]

(34)

From equation (7), we have:

\[
\frac{d\tau_B}{dt} = \sqrt{1 - \frac{v(t)^2}{c^2}} = \frac{1}{\sqrt{1 + \left(\frac{a(t-T)}{c}\right)^2}}
\]

(35)

We have:

\[
\tau_B(t) = \int_{\frac{3T}{4}}^{t} \frac{dt}{\sqrt{1 + \left(\frac{a(t-T)}{c}\right)^2}} + \tau_B\left(\frac{3T}{4}\right) = \frac{c}{a} \left[ \text{arsinh}\left(\frac{a(t - T)}{c}\right) + 4\text{arsinh}\left(\frac{aT}{4c}\right) \right]
\]

(36)

\[
\tau_B(T) = \frac{4c}{a} \text{arsinh}\left(\frac{aT}{4c}\right)
\]

(37)
Summing up

The speed:

\[
v(t) = \frac{at}{\sqrt{1 + \left(\frac{at}{c}\right)^2}} \quad 0 \leq t \leq \frac{T}{4} \tag{38}
\]

\[
v(t) = \frac{a\left(\frac{T}{2} - t\right)}{\sqrt{1 + \left(\frac{a\left(\frac{T}{2} - t\right)}{c}\right)^2}} \quad \frac{T}{4} < t \leq \frac{3T}{4} \tag{39}
\]

\[
v(t) = \frac{a(t - T)}{\sqrt{1 + \left(\frac{a(t - T)}{c}\right)^2}} \quad \frac{3T}{4} < t \leq T \tag{40}
\]

The position:

\[
x(t) = \frac{c^2}{a} \left(\sqrt{1 + \left(\frac{at}{c}\right)^2} - 1\right) \quad 0 \leq t \leq \frac{T}{4} \tag{41}
\]

\[
x(t) = \frac{c^2}{a} \left(2\sqrt{1 + \left(\frac{aT}{4c}\right)^2} - \sqrt{1 + \left(\frac{a\left(\frac{T}{2} - t\right)}{c}\right)^2} - 1\right) \quad \frac{T}{4} < t \leq \frac{3T}{4} \tag{42}
\]

\[
x(t) = \frac{c^2}{a} \left(\sqrt{1 + \left(\frac{a(T - t)}{c}\right)^2} - 1\right) \quad \frac{3T}{4} < t \leq T \tag{43}
\]

The rate of twin B’s clock as observed by twin A:

\[
\frac{d\tau_B}{dt} = \frac{1}{\sqrt{1 + \left(\frac{at}{c}\right)^2}} \quad 0 \leq t \leq \frac{T}{4} \tag{44}
\]

\[
\frac{d\tau_B}{dt} = \frac{1}{\sqrt{1 + \left(\frac{a\left(\frac{T}{2} - t\right)}{c}\right)^2}} \quad \frac{T}{4} < t \leq \frac{3T}{4} \tag{45}
\]

\[
\frac{d\tau_B}{dt} = \frac{1}{\sqrt{1 + \left(\frac{a(t - T)}{c}\right)^2}} \quad \frac{3T}{4} < t \leq T \tag{46}
\]

The proper time of twin B’s clock:

\[
\tau_B(t) = \frac{c}{a} \text{arsinh} \left(\frac{at}{c}\right) \quad 0 \leq t \leq \frac{T}{4} \tag{47}
\]

\[
\tau_B(t) = \frac{c}{a} \left[\text{arsinh} \left(\frac{a\left(\frac{T}{2} - t\right)}{c}\right) + 2\text{arsinh} \left(\frac{aT}{4c}\right)\right] \quad \frac{T}{4} < t \leq \frac{3T}{4} \tag{48}
\]

\[
\tau_B(t) = \frac{c}{a} \left[\text{arsinh} \left(\frac{a(t - T)}{c}\right) + 4\text{arsinh} \left(\frac{aT}{4c}\right)\right] \quad \frac{3T}{4} < t \leq T \tag{49}
\]
2.4 Concrete example

We will use the following units:

- distance: light year \([ly]\)
- time: year \([y]\)
- speed: light year per year \([ly/\text{y}]\)
- acceleration: speed per year \([ly/\text{y}^2]\)

We will calculate the twin 'paradox' scenario in chapter 2.2 with \(L = 10 \text{ly}\) and \(a = c\) per year = \(1\text{ly/\text{y}^2}\).

Equation (18) gives:

\[
T = 2 \sqrt{\frac{L^2}{c^2} + \frac{4L}{a}} = 23.664\text{y}
\]

Using these numbers in equations (38),(39) and (40) and gives the following speed of twin B as a function of the time:

![Figure 1: The speed of twin B in K](image)

Figure 1: The speed of twin B in K
Equations (41), (42) and (43) give the following position of twin B as a function of the time:

![Figure 2: The position of twin B in K](image)

Equations (44), (45) and (46) give the following rate of twin B’s clock as observed by twin A, as a function of the time:

![Figure 3: The rate of twin B’s clock as observed by twin A.](image)
Equations (47), (48) and (49) give the proper time of twin $B$ as a function of the time in $\mathcal{K}$.

Figure 4: The proper time of twin $B$ as a function of the time in $\mathcal{K}$.
3 A twin ’paradox’ simulation

On my homepage [https://paulba.no/](https://paulba.no/) you will find a twin paradox simulation:

**Run the twin ’paradox’ simulation**

The figure below is a picture of the screen when this simulation is run with the same parameters as in chapter 2.4.

![Simulation Screen](image)

Figure 5: A run of the twin paradox simulation with parameters as in chapter 2.4

Compare this to the figures in chapter 2.4.