# The twin 'paradox' calculated by the metric 

Paul B. Andersen

October 12, 2015

## 1 Spacetime interval and proper time

The metric equation is generally:

$$
\begin{equation*}
\mathrm{d} s^{2}=g_{\mu \nu} \mathrm{d} x^{\mu} \mathrm{d} x^{\nu} \tag{1}
\end{equation*}
$$

This is a differential equation. To find the spacetime interval $s$ between two events, one must integrate (1) over a path in spacetime between the events.

When $s^{2}$ is positive, the spacetime interval is space-like, when $s^{2}$ is negative, the spacetime interval is time-like. The interval between two events on the worldline of an object will always be time-like, in which case we will call the interval the proper time of the object.

The differential equation for the proper time can then be written:

$$
\begin{equation*}
\mathrm{d} \tau^{2}=-g_{\mu \nu} \mathrm{d} x^{\mu} \mathrm{d} x^{\nu} \tag{2}
\end{equation*}
$$

In flat spacetime the metric tensor is:

$$
g_{\mu \nu}=\left(\begin{array}{cccc}
-1 & 0 & 0 & 0  \tag{3}\\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

And the equation becomes:

$$
\begin{equation*}
(\mathrm{d} \tau)^{2}=\left(\mathrm{d} x^{0}\right)^{2}-\left(\mathrm{d} x^{1}\right)^{2}-\left(\mathrm{d} x^{2}\right)^{2}-\left(\mathrm{d} x^{3}\right)^{2} \tag{4}
\end{equation*}
$$

Alternatively:

$$
\begin{equation*}
(c \cdot \mathrm{~d} \tau)^{2}=(c \cdot \mathrm{~d} t)^{2}-\mathrm{d} x^{2}-\mathrm{d} y^{2}-\mathrm{d} z^{2} \tag{5}
\end{equation*}
$$

Where $\tau$ is the proper time of some object, while $[t, x, y, z]$ are the coordinates of an inertial frame of reference $\mathcal{K}$, and $c$ is the speed of light in vacuum.

If we use the coordinate time $t$ as parameter, the equation can be written:

$$
\begin{equation*}
\mathrm{d} \tau^{2}=\left(1-\frac{1}{c^{2}}\left[\left(\frac{\mathrm{~d} x}{\mathrm{~d} t}\right)^{2}+\left(\frac{\mathrm{d} y}{\mathrm{~d} t}\right)^{2}+\left(\frac{\mathrm{d} z}{\mathrm{~d} t}\right)^{2}\right]\right) \mathrm{d} t^{2} \tag{6}
\end{equation*}
$$

Which can be simplified to:

$$
\begin{equation*}
\mathrm{d} \tau=\sqrt{1-\frac{v(t)^{2}}{c^{2}}} \mathrm{~d} t \tag{7}
\end{equation*}
$$

where $\vec{v}(t) \underset{\mathcal{K}}{ }\left(\frac{\mathrm{d} x}{\mathrm{~d} t}, \frac{\mathrm{~d} y}{\mathrm{~d} t}, \frac{\mathrm{~d} z}{\mathrm{~d} t}\right)$ and $v(t)^{2}=\vec{v}(t) \cdot \vec{v}(t)=\left(\frac{\mathrm{d} x}{\mathrm{~d} t}\right)^{2}+\left(\frac{\mathrm{d} y}{\mathrm{~d} t}\right)^{2}+\left(\frac{\mathrm{d} z}{\mathrm{~d} t}\right)^{2}$.
The proper time between two events on the object's world line will be:

$$
\begin{equation*}
\tau=\int_{t_{0}}^{t_{1}} \sqrt{1-\frac{v(t)^{2}}{c^{2}}} \mathrm{~d} t \tag{8}
\end{equation*}
$$

Note that $v(t)$ may be any function of $t$.

## 2 The twin 'paradox'

### 2.1 General scenario

Given an inertial frame $\mathcal{K}$ with coordinates $[t, x, y, z]$.
Twin $A$ stays stationary at $x=0$ in $\mathcal{K}$, while twin $B$ starts from $x=0$ when $A$ 's clock shows 0 and travels to some point $x=L$, where she turns around and travels back to $x=0$. She is back at the time $T$ as measured in $\mathcal{K}$. Since twin $A$ is stationary in $\mathcal{K}$, her proper time when twin $B$ is back will be $\tau_{A}=T$.

### 2.2 B travels with constant speed and instant acceleration

We will assume that twin $B$ travels at the constant speed $v$ and turns abruptly around with a brief, very high acceleration $a$ for a very short time $\Delta t$ such that $\lim _{\Delta t \rightarrow 0}(a \Delta t)=2 v$ and thereafter travels back at the constant speed $v$.
There are three events of interest. The coordinates of these events in $\mathcal{K}$ are:

$$
\begin{array}{rlll}
\text { Start } E_{0}: & t=0, & x=0, & y=0, \\
\text { B turns around } E_{1}: & t=\frac{T}{2}, & x=\frac{v T}{2}, & y=0, \\
\text { B is back } E_{2}: & t=T, & x=0, & y=0, \\
z=0 \\
z=0,
\end{array}
$$

The proper times of the twins between the events $E_{0}$ and $E_{2}$ will be:
Twin A:

$$
\begin{equation*}
\tau_{A}=\int_{0}^{T} \mathrm{~d} t=T \tag{9}
\end{equation*}
$$

Twin B:

$$
\begin{equation*}
\tau_{B}=\int_{0}^{\frac{T}{2}} \sqrt{1-\frac{v^{2}}{c^{2}}} \mathrm{~d} t+\int_{\frac{T}{2}}^{T} \sqrt{1-\frac{v^{2}}{c^{2}}} \mathrm{~d} t=T \sqrt{1-\frac{v^{2}}{c^{2}}} \tag{10}
\end{equation*}
$$

Thus:

$$
\begin{equation*}
\tau_{B}=\tau_{A} \sqrt{1-\frac{v^{2}}{c^{2}}} \tag{11}
\end{equation*}
$$

### 2.3 B travels with constant acceleration

Twin $B$ starts from a standstill at the event $E_{0} \overrightarrow{\mathcal{K}}(0,0,0,0)$ and accelerates away with the constant proper acceleration $a$ until the event $E_{1} \overrightarrow{\mathcal{K}}\left(t_{1}, x_{1}, 0,0\right)$ when she reverses the direction of the acceleration and accelerates towards the starting point with constant proper acceleration with magnitude $a$. At the event $E_{2} \underset{\mathcal{K}}{ }\left(t_{2}, L, 0,0\right)$ she will be stationary in $\mathcal{K}$, and starts moving towards the starting point. At the event $E_{3} \overrightarrow{\mathcal{K}}\left(t_{3}, x_{3}, 0,0\right)$ she reverses the direction of acceleration, and accelerates away from the starting point (brakes). She is stationary at the starting point at the event $E_{4} \overrightarrow{\mathcal{K}}$ (T,0,0,0).

Symmetry makes it obvious that $t_{1}=\frac{1}{4} T, t_{2}=\frac{1}{2} T, t_{3}=\frac{3}{4} T$ and $x_{1}=x_{3}=\frac{1}{2} L$.
Let the mass of twin $B$ be $m$. Let $\mathcal{K}^{\prime}$ with coordinates [ $x^{\prime}, y^{\prime}, z^{\prime}, t^{\prime}$ ] be the momentarily co-moving inertial frame of $\operatorname{twin} B$. The $x^{\prime}$-axis of $\mathcal{K}^{\prime}$ is aligned with the $x$-axis of $\mathcal{K}$, and $\mathcal{K}^{\prime}$ is moving at the speed $v$ in $\mathcal{K}$.

Since the accelerating force $F$ must be in the same direction as the velocity of $\mathcal{K}^{\prime}$ in $\mathcal{K}$, the accelerating force must be the same in both frames of reference. The consequence of this is:

$$
\begin{equation*}
\frac{\mathrm{d} p}{\mathrm{~d} t}=\frac{\mathrm{d} p^{\prime}}{\mathrm{d} t^{\prime}}=F \tag{12}
\end{equation*}
$$

Where $p$ is the momentum of twin $B$ in $\mathcal{K}$ and $p^{\prime}$ is the momentum in $\mathcal{K}^{\prime}$. We have $p=\frac{m v}{\sqrt{1-\frac{v^{2}}{c^{2}}}}$ and $\frac{\mathrm{d} p^{\prime}}{\mathrm{d} t^{\prime}}=m a$, where $v$ is the speed of twin $B$ along the $x$-axis of $\mathcal{K}$ and $a$ is the proper acceleration of twin $B$.

Thus:

$$
\begin{equation*}
m \frac{\mathrm{~d}}{\mathrm{~d} t}\left(\frac{v(t)}{\sqrt{1-\frac{v(t)^{2}}{c^{2}}}}\right)=m a \tag{13}
\end{equation*}
$$

Twin B is accelerating outwards to $\frac{\mathrm{L}}{2}, \quad 0 \leq \mathrm{t} \leq \frac{\mathrm{T}}{4}$
We have $v(0)=0$ :

$$
\begin{equation*}
\frac{v(t)}{\sqrt{1-\frac{v(t)^{2}}{c^{2}}}}=\int_{0}^{t} a \mathrm{~d} t=a t \tag{14}
\end{equation*}
$$

Solving this equation with respect to $v$ yields:

$$
\begin{equation*}
v(t)=\frac{a t}{\sqrt{1+\left(\frac{a t}{c}\right)^{2}}} \tag{15}
\end{equation*}
$$

The position $x(t)$ of twin $B$ in $\mathcal{K}$ when $x(0)=0$ is:

$$
\begin{equation*}
x(t)=\int_{0}^{t} v(t) \mathrm{d} t=\frac{c^{2}}{a}\left(\sqrt{1+\left(\frac{a t}{c}\right)^{2}}-1\right) \tag{16}
\end{equation*}
$$

$$
\begin{equation*}
x\left(\frac{T}{4}\right)=\frac{c^{2}}{a}\left(\sqrt{1+\left(\frac{a T}{4 c}\right)^{2}}-1\right)=\frac{L}{2} \tag{17}
\end{equation*}
$$

Solving this equation with respect to $T$ yields:

$$
\begin{equation*}
T=2 \sqrt{\frac{L^{2}}{c^{2}}+\frac{4 L}{a}} \tag{18}
\end{equation*}
$$

From equation (7), we have:

$$
\begin{equation*}
\frac{\mathrm{d} \tau_{B}}{\mathrm{~d} t}=\sqrt{1-\frac{v(t)^{2}}{c^{2}}}=\frac{1}{\sqrt{1+\left(\frac{a t}{c}\right)^{2}}} \tag{19}
\end{equation*}
$$

Since $\tau_{B}(0)=0$, we have:

$$
\begin{align*}
\tau_{B}(t)= & \int_{0}^{t} \frac{\mathrm{~d} t}{\sqrt{1+\left(\frac{a t}{c}\right)^{2}}}=\frac{c}{a} \operatorname{arsinh}\left(\frac{a t}{c}\right)  \tag{20}\\
& \tau_{B}\left(\frac{T}{4}\right)=\frac{c}{a} \operatorname{arsinh}\left(\frac{a T}{4 c}\right) \tag{21}
\end{align*}
$$

Twin B is braking to L, and then accelerating towards $\frac{L}{2}, \quad \frac{T}{4}<\mathrm{t} \leq \frac{3 \mathrm{~T}}{4}$ We have $v\left(\frac{T}{2}\right)=0$ :

$$
\begin{equation*}
\frac{v(t)}{\sqrt{1-\frac{v(t)^{2}}{c^{2}}}}=\int_{\frac{T}{2}}^{t}-a \mathrm{~d} t=a\left(\frac{T}{2}-t\right) \tag{22}
\end{equation*}
$$

Solving this equation with respect to $v$ yields:

$$
\begin{equation*}
v(t)=\frac{a\left(\frac{T}{2}-t\right)}{\sqrt{1+\left(\frac{a\left(\frac{T}{2}-t\right)}{c}\right)^{2}}} \tag{23}
\end{equation*}
$$

The position of twin $B$ in $\mathcal{K}$ is:

$$
\begin{align*}
x(t)=\int_{\frac{T}{4}}^{t} v(t) \mathrm{d} t+x\left(\frac{T}{4}\right) & =\frac{c^{2}}{a}\left(2 \sqrt{1+\left(\frac{a T}{4 c}\right)^{2}}-\sqrt{1+\left(\frac{a\left(\frac{T}{2}-t\right)}{c}\right)^{2}}-1\right)  \tag{24}\\
x\left(\frac{T}{2}\right) & =\frac{2 c^{2}}{a}\left(\sqrt{1+\left(\frac{a T}{4 c}\right)^{2}}-1\right)=L  \tag{25}\\
x\left(\frac{3 T}{4}\right) & =\frac{c^{2}}{a}\left(\sqrt{1+\left(\frac{a T}{4 c}\right)^{2}}-1\right)=\frac{L}{2} \tag{26}
\end{align*}
$$

From equation (7), we have:

$$
\begin{equation*}
\frac{\mathrm{d} \tau_{B}}{\mathrm{~d} t}=\sqrt{1-\frac{v(t)^{2}}{c^{2}}}=\frac{1}{\sqrt{1+\left(\frac{a\left(\frac{T}{2}-t\right)}{c}\right)^{2}}} \tag{27}
\end{equation*}
$$

We have:

$$
\begin{align*}
\tau_{B}(t)=\int_{\frac{T}{4}}^{t} \frac{\mathrm{~d} t}{\sqrt{1+\left(\frac{a\left(\frac{T}{2}-t\right)}{c}\right)^{2}}}+\tau_{B}\left(\frac{T}{4}\right) & =\frac{c}{a}\left[\operatorname{arsinh}\left(\frac{a\left(t-\frac{T}{2}\right)}{c}\right)+2 \operatorname{arsinh}\left(\frac{a T}{4 c}\right)\right]  \tag{28}\\
\tau_{B}\left(\frac{T}{2}\right) & =\frac{2 c}{a} \operatorname{arsinh}\left(\frac{a T}{4 c}\right)  \tag{29}\\
\tau_{B}\left(\frac{3 T}{4}\right) & =\frac{3 c}{a} \operatorname{arsinh}\left(\frac{a T}{4 c}\right) \tag{30}
\end{align*}
$$

Twin B is braking to the starting point, $\quad \frac{3}{4} \mathrm{~T}<\mathrm{t} \leq \mathrm{T}$
We have $v(T)=0$ :

$$
\begin{equation*}
\frac{v(t)}{\sqrt{1-\frac{v(t)^{2}}{c^{2}}}}=\int_{T}^{t} a \mathrm{~d} t=a(t-T) \tag{31}
\end{equation*}
$$

Solving this equation with respect to $v$ yields:

$$
\begin{equation*}
v(t)=\frac{a(t-T)}{\sqrt{1+\left(\frac{a(t-T)}{c}\right)^{2}}} \tag{32}
\end{equation*}
$$

The position of twin $B$ in $\mathcal{K}$ is:

$$
\begin{gather*}
x(t)=\int_{\frac{3 T}{4}}^{t} v(t) \mathrm{d} t+x\left(\frac{3 T}{4}\right)=\frac{c^{2}}{a}\left(\sqrt{1+\left(\frac{a(T-t)}{c}\right)^{2}}-1\right)  \tag{33}\\
x(T)=0 \tag{34}
\end{gather*}
$$

From equation (7), we have:

$$
\begin{equation*}
\frac{\mathrm{d} \tau_{B}}{\mathrm{~d} t}=\sqrt{1-\frac{v(t)^{2}}{c^{2}}}=\frac{1}{\sqrt{1+\left(\frac{a(t-T)}{c}\right)^{2}}} \tag{35}
\end{equation*}
$$

We have:

$$
\begin{align*}
\tau_{B}(t)=\int_{\frac{3 T}{4}}^{t} \frac{\mathrm{~d} t}{\sqrt{1+\left(\frac{a(t-T)}{c}\right)^{2}}}+\tau_{B}\left(\frac{3 T}{4}\right) & =\frac{c}{a}\left[\operatorname{arsinh}\left(\frac{a(t-T)}{c}\right)+4 \operatorname{arsinh}\left(\frac{a T}{4 c}\right)\right]  \tag{36}\\
\tau_{B}(T) & =\frac{4 c}{a} \operatorname{arsinh}\left(\frac{a T}{4 c}\right) \tag{37}
\end{align*}
$$

## Summing up

The speed:

$$
\begin{array}{ll}
v(t)=\frac{a t}{\sqrt{1+\left(\frac{a t}{c}\right)^{2}}} & 0 \leq t \leq \frac{T}{4} \\
v(t)=\frac{a\left(\frac{T}{2}-t\right)}{\sqrt{1+\left(\frac{a\left(\frac{T}{2}-t\right)}{c}\right)^{2}}} & \frac{T}{4}<t \leq \frac{3 T}{4} \\
v(t)=\frac{a(t-T)}{\sqrt{1+\left(\frac{a(t-T)}{c}\right)^{2}}} & \frac{3}{4} T<t \leq T
\end{array}
$$

The position:

$$
\begin{array}{ll}
x(t)=\frac{c^{2}}{a}\left(\sqrt{1+\left(\frac{a t}{c}\right)^{2}}-1\right) & 0 \leq t \leq \frac{T}{4} \\
x(t)=\frac{c^{2}}{a}\left(2 \sqrt{1+\left(\frac{a T}{4 c}\right)^{2}}-\sqrt{1+\left(\frac{a\left(\frac{T}{2}-t\right)}{c}\right)^{2}}-1\right) & \frac{T}{4}<t \leq \frac{3 T}{4} \\
x(t)=\frac{c^{2}}{a}\left(\sqrt{1+\left(\frac{a(T-t)}{c}\right)^{2}}-1\right) & \frac{3}{4} T<t \leq T
\end{array}
$$

The rate of twin B's clock as observed by twin A:

$$
\begin{array}{rll}
\frac{\mathrm{d} \tau_{B}}{\mathrm{~d} t}=\frac{1}{\sqrt{1+\left(\frac{a t}{c}\right)^{2}}} & 0 \leq t \leq \frac{T}{4} \\
\frac{\mathrm{~d} \tau_{B}}{\mathrm{~d} t}=\frac{1}{\sqrt{1+\left(\frac{a\left(\frac{T}{2}-t\right)}{c}\right)^{2}}} & \frac{T}{4}<t \leq \frac{3 T}{4} \\
\frac{\mathrm{~d} \tau_{B}}{\mathrm{~d} t}=\frac{1}{\sqrt{1+\left(\frac{a(t-T)}{c}\right)^{2}}} & \frac{3}{4} T<t \leq T
\end{array}
$$

The proper time of twin B's clock :

$$
\begin{align*}
\tau_{B}(t) & =\frac{c}{a} \operatorname{arsinh}\left(\frac{a t}{c}\right) & 0 \leq t \leq \frac{T}{4}  \tag{47}\\
\tau_{B}(t) & =\frac{c}{a}\left[\operatorname{arsinh}\left(\frac{a\left(t-\frac{T}{2}\right)}{c}\right)+2 \operatorname{arsinh}\left(\frac{a T}{4 c}\right)\right] & \frac{T}{4}<t \leq \frac{3 T}{4} \\
\tau_{B}(t) & =\frac{c}{a}\left[\operatorname{arsinh}\left(\frac{a(t-T)}{c}\right)+4 \operatorname{arsinh}\left(\frac{a T}{4 c}\right)\right] & \frac{3}{4} T<t \leq T \tag{48}
\end{align*}
$$

### 2.4 Concrete example

We will use the following units:

| distance: | light year | $[l y]$ |
| :--- | :--- | :--- |
| time: | year | $[y]$ |
| speed: | light year per year | $\left[\frac{l y}{y}\right]$ |
| acceleration: | speed per year | $\left[\frac{l y}{y^{2}}\right]$ |

We will calculate the twin 'paradox' scenario in chapter 2.2 with $L=10 l y$ and $a=c$ per year $=1 \frac{l y}{y^{2}}$.

Equation (18) gives:

$$
T=2 \sqrt{\frac{L^{2}}{c^{2}}+\frac{4 L}{a}}=23.664 y
$$

Using these numbers in equations (38),(39) and (40) and gives the following speed of twin $B$ as a function of the time:


Figure 1: The speed of twin $B$ in $\mathcal{K}$

Equations (41),(42) and (43) give the following position of twin $B$ as a function of the time:


Figure 2: The position of twin $B$ in $\mathcal{K}$

Equations (44),(45) and (46) give the following rate of twin $B$ 's clock as observed by twin $A$, as a function of the time:


Figure 3: The rate of twin $B^{\prime}$ s clock as observed by twin $A$.

Equations (47),(48) and (49) give the proper time of twin $B$ as a function of the time in $\mathcal{K}$.


Figure 4: The proper time of twin $B$ as a function of the time in $\mathcal{K}$

## 3 A twin 'paradox' simulation

On my homepage https://paulba.no/ you will find a twin paradox simulation:

## Run the twin 'paradox' simulation

The figure below is a picture of the screen when this simulation is run with the same parameters as in chapter 2.4.


Figure 5: A run of the twin paradox simulation with parameters as in chapter 2.4
Compare this to the figures in chapter 2.4.

