The twin 'paradox' calculated by Doppler shift

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The scenario

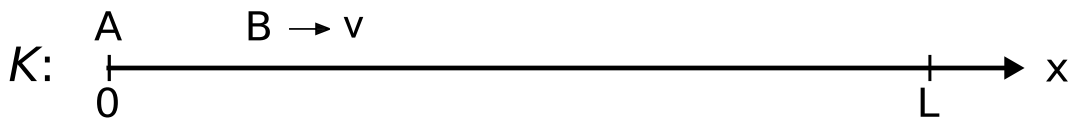

Figure 1: *Frame K*

Twin A stays stationary at $x = 0$ in an inertial frame K, while twin B travels at the speed v to $x = L$, where she turns around and travels back to A at the speed v. Both A and B emit EM-pulses (or light flashes) with the pulse repetition frequency f, and both count the received pulses. When they are back together, both must have counted all the pulses emitted by the other twin.

Observations made by A:

When B is going out, A will receive the pulses with the Doppler shifted frequency $f_1 = f\sqrt{\frac{c-v}{c+v}}$. She will observe this frequency for the time it takes B to reach the turning point at some distance L, plus the time it takes for the light to reach her from the turning point: $t_{a1} = \frac{L}{v} + \frac{L}{c}$. Thereafter, A will observe the Doppler shifted frequency $f_2 = f\sqrt{\frac{c+v}{c-v}}$ for the time $t_{a2} = \frac{L}{v} - \frac{L}{c}$.

A will receive the total number of pulses:

$$\begin{aligned} N_A &= (f_1 \cdot t_{a1}) + (f_2 \cdot t_{a2}) \\ &= f\sqrt{\frac{c-v}{c+v}}\left(\frac{L}{v} + \frac{L}{c}\right) + f\sqrt{\frac{c+v}{c-v}}\left(\frac{L}{v} - \frac{L}{c}\right) \\ &= f\frac{L}{vc}\sqrt{(c-v)(c+v)} + f\frac{L}{vc}\sqrt{(c+v)(c-v)} \\ &= \frac{2L}{v}\sqrt{1 - \frac{v^2}{c^2}} \cdot f \end{aligned}$$

Measured on twin A's clock the duration of the journey will be $T_A = \frac{2L}{v}$, thus:

$$N_A = T_A \sqrt{1 - \frac{v^2}{c^2}} \cdot f \tag{1}$$

Let T_B be the proper time shown by B's clock when she returns. The number of pulses received by A must obviously be equal to the number of pulses emitted by B, and since B emits $T_B \cdot f$ pulses, B's clock when she returns must show:

$$T_B = T_A \sqrt{1 - \frac{v^2}{c^2}} \tag{2}$$

Observations made by B:

Let T_B be the proper time measured by B's clock when she returns. B will obviously use the same time on the journey out and the journey back. When going out B will receive the pulses from A with the Doppler shifted frequency $f_1 = f\sqrt{\frac{c-v}{c+v}}$ for the time $t_{b1} = \frac{T_B}{2}$. When B turns around, she will immediately receive the pulses from A with the frequency $f_2 = f\sqrt{\frac{c+v}{c-v}}$ for the time $t_{b2} = \frac{T_B}{2}$.

B will receive the total number of pulses:

$$\begin{aligned} N_B &= (f_1 \cdot t_{b1}) + (f_2 \cdot t_{b2}) \\ &= f\sqrt{\frac{c-v}{c+v}} \cdot \frac{T_B}{2} + f\sqrt{\frac{c+v}{c-v}} \cdot \frac{T_B}{2} \\ &= f\frac{T_B\,(c-v)}{2\sqrt{c^2-v^2}} + f\frac{T_B\,(c+v)}{2\sqrt{c^2-v^2}} = \frac{T_B}{\sqrt{1-\frac{v^2}{c^2}}}\; f \end{aligned}$$

$$N_B = \frac{T_B}{\sqrt{1-\frac{v^2}{c^2}}}\; f \tag{3}$$

The number of pulses received by B must obviously be equal to the number of pulses emitted by A, and since A has emitted $T_A \cdot f$ pulses, A's clock when B returns must show:

$$T_A = \frac{T_B}{\sqrt{1-\frac{v^2}{c^2}}} \tag{4}$$

Conclusion

A and B will, based on their own observation, agree that $T_A = \frac{T_B}{\sqrt{1-\frac{v^2}{c^2}}}$.

Twin A ages more than twin B.