The twin 'paradox' calculated by Doppler shift

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May 16, 2015

The scenario

Figure 1: Frame $K$

Twin $A$ stays stationary at $x = 0$ in an inertial frame $K$, while twin $B$ travels at the speed $v$ to $x = L$, where she turns around and travels back to $A$ at the speed $v$. Both $A$ and $B$ emit EM-pulses (or light flashes) with the pulse repetition frequency $f$, and both count the received pulses. When they are back together, both must have counted all the pulses emitted by the other twin.

Observations made by $A$:

When $B$ is going out, $A$ will receive the pulses with the Doppler shifted frequency $f_1 = f \sqrt{\frac{c-v}{c+v}}$. She will observe this frequency for the time it takes $B$ to reach the turning point at some distance $L$, plus the time it takes for the light to reach her from the turning point: $t_{a1} = \frac{L}{v} + \frac{L}{c}$. Thereafter, $A$ will observe the Doppler shifted frequency $f_2 = f \sqrt{\frac{c+v}{c-v}}$ for the time $t_{a2} = \frac{L}{v} - \frac{L}{c}$.

$A$ will receive the total number of pulses:

$$N_A = (f_1 \cdot t_{a1}) + (f_2 \cdot t_{a2})$$

$$= f \sqrt{\frac{c-v}{c+v}} \left( \frac{L}{v} + \frac{L}{c} \right) + f \sqrt{\frac{c+v}{c-v}} \left( \frac{L}{v} - \frac{L}{c} \right)$$

$$= f \frac{L}{vc} \sqrt{(c-v)(c+v)} + f \frac{L}{vc} \sqrt{(c+v)(c-v)}$$

$$= \frac{2L}{v} \sqrt{1 - \frac{v^2}{c^2}} \cdot f$$
Measured on twin $A$’s clock the duration of the journey will be $T_A = \frac{2L}{v}$, thus:

$$N_A = T_A \sqrt{1 - \frac{v^2}{c^2}} \cdot f$$ \hspace{1cm} (1)

Let $T_B$ be the proper time shown by $B$’s clock when she returns. The number of pulses received by $A$ must obviously be equal to the number of pulses emitted by $B$, and since $B$ emits $T_B \cdot f$ pulses, $B$’s clock when she returns must show:

$$T_B = T_A \sqrt{1 - \frac{v^2}{c^2}}$$ \hspace{1cm} (2)

**Observations made by $B$:**

Let $T_B$ be the proper time measured by $B$’s clock when she returns. $B$ will obviously use the same time on the journey out and the journey back. When going out $B$ will receive the pulses from $A$ with the Doppler shifted frequency $f_1 = f \sqrt{\frac{c-v}{c+v}}$ for the time $t_{b1} = \frac{T_B}{2}$. When $B$ turns around, she will immediately receive the pulses from $A$ with the frequency $f_2 = f \sqrt{\frac{c+v}{c-v}}$ for the time $t_{b2} = \frac{T_B}{2}$.

$B$ will receive the total number of pulses:

$$N_B = (f_1 \cdot t_{b1}) + (f_2 \cdot t_{b2})$$

$$= f \sqrt{\frac{c-v}{c+v}} \cdot \frac{T_B}{2} + f \sqrt{\frac{c+v}{c-v}} \cdot \frac{T_B}{2}$$

$$= f \frac{T_B (c-v)}{2\sqrt{c^2-v^2}} + f \frac{T_B (c+v)}{2\sqrt{c^2-v^2}} = \frac{T_B}{\sqrt{1 - \frac{v^2}{c^2}}} f$$

$$N_B = \frac{T_B}{\sqrt{1 - \frac{v^2}{c^2}}} f$$ \hspace{1cm} (3)

The number of pulses received by $B$ must obviously be equal to the number of pulses emitted by $A$, and since $A$ has emitted $T_A \cdot f$ pulses, $A$’s clock when $B$ returns must show:

$$T_A = \frac{T_B}{\sqrt{1 - \frac{v^2}{c^2}}}$$ \hspace{1cm} (4)

**Conclusion**

$A$ and $B$ will, based on their own observation, agree that $T_A = \frac{T_B}{\sqrt{1 - \frac{v^2}{c^2}}}$.

Twin $A$ ages more than twin $B$. 

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