Standing Waves

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1 Introduction

This article shows the mathematics behind the simulation **StandingWaves**

2 Standing waves

2.1 The incident wave

A plane, linearly polarised electromagnetic wave propagating in the positive z-direction can be written:

$$E_{ix}(t,z) = E_0 \sin(\omega t - kz) \quad \text{where } k = \frac{2\pi}{\lambda} = \frac{\omega}{c} \text{ in vacuum} \tag{1}$$

From Maxwell's equations we have $\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$.
With $\vec{E} = [E_{ix}, 0, 0]$ this simplifies to:

$$\frac{\partial B_{iy}}{\partial t} = -\frac{\partial E_{ix}}{\partial z} = -\frac{\partial}{\partial z}\left(E_0 \sin(\omega t - kz)\right) = E_0 k \cos(\omega t - kz) \tag{2}$$

Integration yields:

$$B_{iy} = \frac{E_0 k}{\omega} \sin(\omega t - kz) = \frac{E_0}{c} \sin(\omega t - kz) \tag{3}$$

or

$$H_{iy}(t,z) = H_0 \sin(\omega t - kz) \tag{4}$$

where $H_0 = \frac{E_0}{Z_0}$, $Z_0 = \mu_0 c = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{1}{\varepsilon_0 c}$, the impedance of free space.

2.2 The reflected wave

Let us assume that the wave is reflected off a conducting mirror in the x-y plane at $z = 0$.
We know that the amplitude, frequency and wavelength must be the same for the reflected wave as for the incident wave, but it is propagating in the negative z-direction. So the reflected wave must be of the form:

$$E_{rx}(t,z) = E_0 \sin(\omega t + kz + \theta_E) \tag{5}$$

$$H_{ry}(t,z) = H_0 \sin(\omega t + kz + \theta_H) \tag{6}$$

θ_E and θ_H are given by the boundary conditions at the mirror, which are:

E-field zero at the mirror: $E_{rx}(t,0) = -E_{ix}(t,0)$, yielding $\theta_E = \pi$, so:

$$E_{rx}(t,z) = -E_0 \sin(\omega t + kz) \tag{7}$$

H-field the same for both waves at the mirror:$H_{ry}(t,0) = H_{iy}(t,0) =$, yielding $\theta_H = 0$, so:

$$H_{ry}(t,z) = H_0 \sin(\omega t + kz) \tag{8}$$

2.3 The standing wave

The standing wave is the interference between the incident and reflected wave:

$$E_{sx}(t,z) = E_{ix}(t,z) + E_{rx}(t,z) = E_0 \sin(\omega t - kz) - E_0 \sin(\omega t + kz) \tag{9}$$

$$H_{sy}(t,z) = H_{iy}(t,z) + H_{ry}(t,z) = H_0 \sin(\omega t - kz) + H_0 \sin(\omega t + kz) \tag{10}$$

Using the equation $\sin(\varphi_1 \pm \varphi_2) = \sin\varphi_1 \cos\varphi_2 \pm \cos\varphi_1 \sin\varphi_2$ yields:

$$E_{sx}(t,z) = -2E_0 \cos(\omega t) \sin(kz) \tag{11}$$

$$H_{sy}(t,z) = 2H_0 \sin(\omega t) \cos(kz) \tag{12}$$

The phase difference between the electric and magnetic field is $\pi/2$ both in time and space.

3 Energy density in the waves

3.1 Energy density in a travelling wave

The energy density in an electric field is: $\eta_E = \dfrac{\varepsilon_0 \vec{E}\cdot\vec{E}}{2} = \dfrac{\varepsilon_0 |\vec{E}|^2}{2}$

The energy density in a magnetic field is: $\eta_M = \dfrac{\mu_0 \vec{H}\cdot\vec{H}}{2} = \dfrac{\mu_0 |\vec{H}|^2}{2}$

In a travelling wave we have: $|\vec{H}| = \dfrac{|\vec{E}|}{Z_0}$, so we can write the energy density in the magnetic field:

$$\eta_M = \frac{\mu_0 |\vec{H}|^2}{2} = \frac{\mu_0 |\vec{E}|^2}{2Z_0^2} = \frac{\varepsilon_0 |\vec{E}|^2}{2} = \eta_E \tag{13}$$

In a travelling wave the energy densities of the magnetic and electric field are always equal, so the energy density in the wave is:

$$\eta_{EM} = \varepsilon_0 |\vec{E}|^2 \tag{14}$$

The energy density in the incident wave:

$$E_{ix}(t,z) = E_0 \sin(\omega t - kz) \tag{15}$$

$$H_{iy}(t,z) = H_0 \sin(\omega t - kz) \tag{16}$$

then becomes:

$$\eta_i(t,x) = \varepsilon_0 |\vec{E}(t,x)|^2 = \varepsilon_0 E_0^2 \sin^2(\omega t - kz) = \frac{\varepsilon_0 E_0^2}{2}(1 - \cos(2\omega t - 2kz)) \tag{17}$$

The energy density in the reflected wave:

$$E_{rx}(t,z) = -E_0 \sin(\omega t + kz) \tag{18}$$

$$H_{ry}(t,z) = H_0 \sin(\omega t + kz) \tag{19}$$

becomes:

$$\eta_r(t,x) = \frac{\varepsilon_0 E_0^2}{2}(1 - \cos(2\omega t + 2kz)) \tag{20}$$

3.2 Energy density in the standing wave

Since the electric and magnetic fields are not in phase in the standing wave, the energy densities of the electric and magnetic fields are not generally equal. We have to calculate each of them independently.

The standing wave is:

$$E_{sx}\left(t,z\right) = -2E_0 \cos\left(\omega t\right) \sin\left(kz\right) \tag{21}$$

$$H_{sy}\left(t,z\right) = 2H_0 \sin\left(\omega t\right) \cos\left(kz\right) \tag{22}$$

where $H_0 = \dfrac{E_0}{Z_0}$

The energy density in the electric field is:

$$\eta_E = \frac{\varepsilon_0 |\vec{E}|^2}{2} = 2\varepsilon_0 E_0^2 \cos^2\left(\omega t\right) \sin^2\left(kz\right) \tag{23}$$

The energy density in the magnetic field is:

$$\eta_M = \frac{\mu_0 |\vec{H}|^2}{2} = 2\mu_0 H_0^2 \sin^2\left(\omega t\right) \cos^2\left(kz\right) = 2\varepsilon_0 E_0^2 \sin^2\left(\omega t\right) \cos^2\left(kz\right) \tag{24}$$

The energy density in the standing wave becomes:

$$\eta_{EM} = \eta_E + \eta_M \tag{25}$$

$$\eta_{EM} = 2\varepsilon_0 E_0^2 \left(\cos^2\left(\omega t\right) \sin^2\left(kz\right) + \sin^2\left(\omega t\right) \cos^2\left(kz\right)\right) = 2\varepsilon_0 E_0^2 \left(1 - \cos\left(2\omega t\right) \sin\left(2kz\right)\right) \tag{26}$$