What GR predicts for the Pound - Rebka experiment

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1 Introduction

The Pound-Rebka experiment is one of the classical tests of The General Theory of Relativity (GR). It measures the gravitational frequency shift of a γ-photon in a 22.56 meter high tower.

Pound & Rebka's original papers

We will calculate what GR predicts for the experiment.

2 Calculation of the prediction

2.1 Precise calculation

The Schwarzschild metric for a clock which is stationary relative to the Earth $\left(\frac{\mathrm{d}r}{\mathrm{d}t} = 0 \text{ and } \frac{\mathrm{d}\theta}{\mathrm{d}t} = 0\right)$ is:

$$\mathrm{d}\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right)\mathrm{d}t^2 - \frac{r^2 \sin^2\theta}{c^2}\,\mathrm{d}\varphi^2 \tag{1}$$

or

$$\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 = 1 - \frac{2GM}{c^2 r} - \frac{r^2 \sin^2\theta}{c^2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 \tag{2}$$

where:

- $\boldsymbol{\tau}$ is the proper time
- $\boldsymbol{t}$ is the Schwarzschild temporal coordinate
- $\boldsymbol{r}$ is the Schwarzschild radial coordinate
- $\boldsymbol{\varphi}$ is the Schwarzschild longitude coordinate
- $\boldsymbol{\theta}$ is the colatitude (angle from north)
- $\boldsymbol{G}$ is the gravitational constant
- $\boldsymbol{M}$ is the mass of the Earth
- $\boldsymbol{c}$ is the speed of light in vacuum

The Pound & Rebka experiment was performed at Harvard University, Cambridge, Massachusetts, at latitude $42^o\ 22'\ 30"N$. So $\theta = 47.625^o$, and $\sin^2\theta \approx 0.55$.
So at Harvard the rate of the clock compared to the Schwarzschild temporal coordinate t is:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{1 - \frac{2GM}{c^2 r} - \frac{0.55 \cdot r^2\omega^2}{c^2}} \tag{3}$$

where $\omega = \dfrac{\mathrm{d}\varphi}{\mathrm{d}t}$ is the angular velocity of the Earth.

If we have one clock at the geoid, $r = R$, and one clock at an altitude h above it, $r = R + h$, the rate of the latter compared to the former will be:

$$\frac{\mathrm{d}\tau_1}{\mathrm{d}\tau_2} = \sqrt{\frac{1 - \frac{2GM}{c^2(R+h)} - \frac{0.55\cdot(R+h)^2\omega^2}{c^2}}{1 - \frac{2GM}{c^2R} - \frac{0.55\cdot R^2\omega^2}{c^2}}} \tag{4}$$

If a photon is emitted from the upper clock and detected at the lower clock, the relative frequency change will be:

$$\frac{\Delta\nu}{\nu} = \sqrt{\frac{1 - \frac{2GM}{c^2(R+h)} - \frac{0.55\cdot(R+h)^2\omega^2}{c^2}}{1 - \frac{2GM}{c^2R} - \frac{0.55\cdot R^2\omega^2}{c^2}}} - 1 \tag{5}$$

Using these values for the constants:

$GM = 3.986004418 \cdot 10^{14}\ \frac{m^3}{s^2}$
$c = 299792458\ \frac{m}{s}$
$R = 6368.5\ km$, radius or the Earth at 42.3^oN latitude
$h = 22.56\ m$
$\omega = 7.292116 \cdot 10^{-5}\ \frac{rad}{s}$

yields:

$$\frac{\Delta\nu}{\nu} = 2.494 \cdot 10^{-15} \tag{6}$$

2.2 Approximated calculation

Since the term $\frac{GM}{c^2R} \approx 1.4 \cdot 10^{-9} \ll 1$, and the term $\frac{0.55\cdot R^2\omega^2}{c^2} \approx 1.3 \cdot 10^{-12} \ll 1$, an approximation of equation (5) is:

$$\frac{\Delta\nu}{\nu} \approx \left(\frac{GM}{c^2R} - \frac{GM}{c^2\,(R+h)}\right) - \left(\frac{0.55\cdot(R+h)^2\,\omega^2}{2c^2} - \frac{0.55\cdot R^2\omega^2}{2c^2}\right) \tag{7}$$

$$= \frac{GMh}{c^2R^2}\left(\frac{1}{1+\frac{h}{R}}\right) - \frac{0.55\cdot\omega^2 Rh}{c^2}\left(1 + \frac{h}{2R}\right) \tag{8}$$

Since $\frac{h}{R} \approx 3.5 \cdot 10^{-6} \ll 1$ and $\frac{GM}{R^2} = g$, equation (7) can be simplified to:

$$\frac{\Delta\nu}{\nu} \approx \frac{gh}{c^2} - \frac{0.55 \cdot \omega^2 Rh}{c^2} \tag{9}$$

The first term is what we loosely could call the "gravitational term", while the second term is the "speed term". If we use the value for g at the latitude 42.3^o, $g = 9.804\ \frac{m}{s^2}$, the values of the terms are: $\frac{gh}{c^2} = 2.46 \cdot 10^{-15}$ and $\frac{0.55 \cdot \omega^2 Rh}{c^2} = 4.67 \cdot 10^{-18}$.
If we compare these values with the more precise value in equation (6), we note that the "speed term" is less than the error in the "gravitational term". It is therefore no point in including this term in the rather crude approximation in equation (9).
The "classical" approximated expression is:

$$\frac{\Delta\nu}{\nu} \approx \frac{gh}{c^2} = 2.46 \cdot 10^{-15} \tag{10}$$

3 Comparison with the measurements

The measurements in the Pound - Rebka experiment were done by comparing the difference in frequency shifts when the source was at the top of the tower, and when it was at the bottom. This means that the measured shift is a combination of a blue shift and a red shift, and $\frac{\Delta\nu_{tot}}{\nu} = 2\frac{\Delta\nu}{\nu}$.

So the precise prediction of GR (eq. 6) is:

$$\frac{\Delta\nu_{tot}}{\nu} = 4.988 \cdot 10^{-15} \tag{11}$$

and the approximated prediction (eq. 10) is:

$$\frac{\Delta\nu_{tot}}{\nu} = 4.92 \cdot 10^{-15} \tag{12}$$

which is exactly the same value as Pound & Rebka used for the prediction.

Pound & Rebka measured:

$$\frac{\Delta\nu_{tot}}{\nu} = (5.13 \pm 0.51) \cdot 10^{-15} \tag{13}$$

So both the predictions (11) and (12) are well within the error bars of the measured values. It is however interesting to notice that the more precise prediction (11) is closer to the measured value than is the less precise prediction (12).