An illustration of mutual time dilation

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The scenario

- Let’s have two clocks which are synchronized according to Einstein’s procedure in each of two inertial frames of reference.
- Let the clocks be a proper distance $d$ from each other in their respective frames.
- Let the frames move with the relative speed $v$.

![Figure 1: The frames of reference](image)

There are three events of interest:

**Event $E_1$:** clock $A$ and clock $A'$ are adjacent

**Event $E_2$:** clock $A$ and clock $B'$ are adjacent

**Event $E_3$:** clock $B$ and clock $A'$ are adjacent

Calculation of what the clocks will show at the events

**Event $E_1$:**

$A$ shows $t_1 = 0$, $A'$ shows $t'_1 = 0$ (the clocks are set thus)

**Event $E_2$:**

In frame $K'$, $A$ will be at the position $-d$ when $B'$ shows $t'_2 = \frac{d}{v}$

According to the Lorentz transform, $A$ shows:

$$ t_2 = \frac{d + \frac{dv}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{d}{c} \sqrt{1 - \frac{v^2}{c^2}} \tag{1} $$
Event $E_3$:  
In frame $K$, $A'$ will be at the position $d$ when $B$ shows $t_3 = \frac{d}{v}$

According to the Lorentz transform, $A'$ shows:

$$t'_3 = \frac{d - \frac{dv}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{d}{v} \sqrt{1 - \frac{v^2}{c^2}}$$  \hspace{1cm} (2)

Summing up, the readings of the clocks will be:

At event $E_1$:  
$A$ shows $t_1 = 0$  
$A'$ shows $t'_1 = 0$

At event $E_2$:  
$A$ shows $t_2 = \frac{d}{v} \sqrt{1 - \frac{v^2}{c^2}}$  
$B$ shows $t'_2 = \frac{d}{v}$

At event $E_3$:  
$B$ shows $t_3 = \frac{d}{v}$  
$A'$ shows $t'_3 = \frac{d}{v} \sqrt{1 - \frac{v^2}{c^2}}$

The symmetry is obvious.

Which clock is running slow or fast relative to which?

The answer depends on how the clocks are compared!

In frame $K$ we can measure the rate $\frac{dt'}{dt}$ of the moving clock $A'$ by comparing the reading of $A'$ with the readings of the two clocks $A$ and $B$ as it passes them:

$$\frac{dt'}{dt} = \frac{(t'_3 - t'_1)}{(t_3 - t_1)} = \sqrt{1 - \frac{v^2}{c^2}}$$  \hspace{1cm} (3)

Conclusion #1: Clock $A'$ runs slow as observed in frame $K$

In frame $K'$ we can measure the rate $\frac{dt}{dt'}$ of the moving clock $A$ by comparing the reading of $A$ with the readings of the two clocks $A'$ and $B'$ as it passes them:

$$\frac{dt}{dt'} = \frac{(t_2 - t_1)}{(t'_2 - t'_1)} = \sqrt{1 - \frac{v^2}{c^2}}$$  \hspace{1cm} (4)

Conclusion #2: Clock $A$ runs slow as observed in frame $K'$

This is what is meant by mutual time dilation.

Conclusion #1 does not contradict conclusion #2 because the temporal interval between different sets of events are compared.
But we can draw more conclusions

We can measure the rate $R'$ at which an observer in $K'$ will see the co-ordinate time of $K$ runs by reading the clocks $A$ and $B$ as they pass clock $A'$:

$$R' = \frac{(t_3 - t_1)}{(t'_3 - t'_1)} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

(5)

**Conclusion #3: The co-ordinate time of frame $K$ runs fast as observed in frame $K'$**

We can measure the rate $R$ at which an observer in $K$ will see the co-ordinate time of $K'$ runs by reading the clocks $A'$ and $B'$ as they pass clock $A$:

$$R = \frac{(t'_2 - t'_1)}{(t_2 - t_1)} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

(6)

**Conclusion #4: The co-ordinate time of frame $K'$ runs fast as observed in frame $K$**

There is nothing contradictory between conclusion #3 and #4 either.

It is in fact conclusions #1 and #3 and conclusions #2 and #4 respectively that compare the temporal interval between the same sets of events.