An illustration of mutual time dilation

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The scenario

- Let's have two clocks which are synchronized according to Einstein's procedure in each of two inertial frames of reference.
- Let the clocks be a proper distance d from each other in their respective frames.
- Let the frames move with the relative speed v.

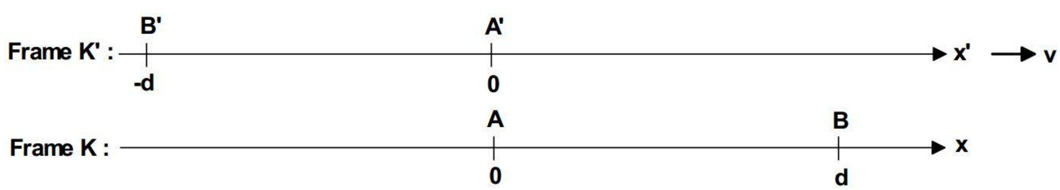

Figure 1: *The frames of reference*

There are three events of interest:

Event E_1: clock A and clock A' are adjacent

Event E_2: clock A and clock B' are adjacent

Event E_3: clock B and clock A' are adjacent

Calculation of what the clocks will show at the events

Event E_1:
A shows $t_1 = 0$, A' shows $t'_1 = 0$ (the clocks are set thus)

Event E_2:
In frame K', A will be at the position $-d$ when B' shows $t'_2 = \frac{d}{v}$

According to the Lorentz transform, A shows:

$$t_2 = \frac{\dfrac{d}{v} + \dfrac{-dv}{c^2}}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \frac{d}{v}\sqrt{1 - \frac{v^2}{c^2}} \tag{1}$$

Event E_3:
In frame K, A' will be at the position d when B shows $t_3 = \frac{d}{v}$

According to the Lorentz transform, A' shows:

$$t'_3 = \frac{\dfrac{d}{v} - \dfrac{dv}{c^2}}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \frac{d}{v}\sqrt{1 - \frac{v^2}{c^2}} \tag{2}$$

Summing up, the readings of the clocks will be:

At event E_1: A shows $t_1 = 0$ A' shows $t'_1 = 0$

At event E_2: A shows $t_2 = \frac{d}{v}\sqrt{1 - \frac{v^2}{c^2}}$ B' shows $t'_2 = \frac{d}{v}$

At event E_3: B shows $t_3 = \frac{d}{v}$ A' shows $t'_3 = \frac{d}{v}\sqrt{1 - \frac{v^2}{c^2}}$

The symmetry is obvious.

Which clock is running slow or fast relative to which?

The answer depends on how the clocks are compared!

In frame K we can measure the rate $\frac{\mathrm{d}t'}{\mathrm{d}t}$ of the moving clock A' by comparing the reading of A' with the readings of the *two* clocks A and B as it passes them:

$$\frac{\mathrm{d}t'}{\mathrm{d}t} = \frac{(t'_3 - t'_1)}{(t_3 - t_1)} = \sqrt{1 - \frac{v^2}{c^2}} \tag{3}$$

Conclusion #1: ***Clock A′ runs slow as observed in frame K***

In frame K' we can measure the rate $\frac{\mathrm{d}t}{\mathrm{d}t'}$ of the moving clock A by comparing the reading of A with the readings of the *two* clocks A' and B' as it passes them:

$$\frac{\mathrm{d}t}{\mathrm{d}t'} = \frac{(t_2 - t_1)}{(t'_2 - t'_1)} = \sqrt{1 - \frac{v^2}{c^2}} \tag{4}$$

Conclusion #2: ***Clock A runs slow as observed in frame K′***

This is what is meant by *mutual time dilation.*

Conclusion #1 does not contradict conclusion #2 because the temporal interval between different sets of events are compared.

But we can draw more conclusions

We can measure the rate R' at which an observer in K' will see the co-ordinate time of K runs by reading the clocks A and B as they pass clock A':

$$R' = \frac{(t_3 - t_1)}{(t'_3 - t'_1)} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5}$$

Conclusion #3: ***The co-ordinate time of frame K runs fast as observed in frame K′***

We can measure the rate R at which an observer in K will see the co-ordinate time of K' runs by reading the clocks A' and B' as they pass clock A:

$$R = \frac{(t'_2 - t'_1)}{(t_2 - t_1)} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{6}$$

Conclusion #4: ***The co-ordinate time of frame K′ runs fast as observed in frame K***

There is nothing contradictory between conclusion #3 and #4 either.

It is in fact conclusions #1 and #3 and conclusions #2 and #4 respectively that compare the temporal interval between the same sets of events.