What GR predicts for the perihelion advance of planets

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1 Simulation of the perihelion advance of planets

The motions of the planets in the Solar system are simulated. Each planet is simulated as a two body problem, planet - Sun. The Newtonian gravitational acceleration is replaced by the post-Newtonian approximation of the acceleration predicted by The General Theory of Relativity [GR].

For each planet is the advance of the perihelion measured to show what GR predicts the rate of perihelion advance would be if the only existing bodies in the Universe were the planet and the Sun.

The planets' initial positions and velocities in the solar system are as they were at Epoch J2000. The data to find these positions and velocities are:

The mass	$\boldsymbol{m}$	where m includes the mass of the planet's moons
The orbital period	$\boldsymbol{P}$	
The semi major axis	$\boldsymbol{a}$	half the distance between the planet at perihelion and aphelion
The perihelion distance	$\boldsymbol{a_p}$	distance Sun - planet at perihelion
The inclination	$\boldsymbol{i}$	to the ecliptic plane
The mean anomaly	$\boldsymbol{M}$	at Epoch J2000
Longitude of ascending node	$\boldsymbol{\Omega}$	at Epoch J2000
Argument of perihelion	$\boldsymbol{\omega}$	at Epoch J2000

The numerical values of the orbital elements are from [3]. Solar mass = $1.98847 \cdot 10^{30}$ kg.

Planet	**Mercury**	**Venus**	**Earth**	**Mars**	**Jupiter**	**Saturn**	**Uranus**	**Neptun**
$\boldsymbol{m}$ $[\boldsymbol{kg}]$	3.301132E23	4.867453E24	6.045809E24	6.417094E24	1.898575E27	5.684754E26	8.682160E25	1.024338E26
$\boldsymbol{P}$ $[\boldsymbol{days}]$	87.969116	224.7016	365.256363	686.9816	4332.589	10759.22	30685.4	60189.0
$\boldsymbol{a}$ $[\mathbf{10^{10}}\boldsymbol{m}]$	5.7909082898	10.820860089	14.959802296	22.793918518	77.829836219	142.93940698	287.50386090	450.44497616
$\boldsymbol{a_p}$ $[\mathbf{10^{10}}\boldsymbol{m}]$	4.6001136690	10.747582129	14.709844432	20.664951765	74.055250620	134.99938841	274.16908047	446.18568243
$\boldsymbol{i}$ $[\ ^o]$	7.00499^o	3.39466^o	0.00000^o	1.84973^o	1.30327^o	2.48888^o	0.77320^o	1.76995^o
$\boldsymbol{M}$ $[\ ^o]$	174.796^o	50.115^o	358.617^o	19.3870^o	20.020^o	317.020^o	142.2386^o	256.228^o
$\boldsymbol{\Omega}$ $[\ ^o]$	48.331^o	76.680^o	-11.26064^o	49.558^o	49.558^o	113.665^o	74.006^o	131.784^o
$\boldsymbol{\omega}$ $[\ ^o]$	29.124^o	54.884^o	114.20783^o	286.502^o	273.867^o	339.392^o	96.999^o	276.336^o

Table 1: Planet data at Epoch J2000

On my homepage https://paulba.no/ you will find the perihelion advance simulation: Run the simulation

2 The post-Newtonian approximation of the gravitational acceleration predicted by GR

The post-Newtonian acceleration of a planet is according to [1], equation (3.11):

$$\frac{\mathrm{d}^2\vec{r}}{c^2\mathrm{d}t^2} = -\frac{\mu}{r^3}\vec{r} + \frac{\mu}{r^3}\left[\left(4\frac{\mu}{r} - \frac{v^2}{c^2}\right)\vec{r} + 4\frac{(\vec{r}\cdot\vec{v})\,\vec{v}}{c^2}\right] \tag{1}$$

With $\mu = \frac{GM}{c^2}$ we get:

$$\frac{\mathrm{d}^2\vec{r}}{c^2\mathrm{d}t^2} = -\frac{GM}{r^3c^2}\vec{r} + \frac{GM}{r^3c^2}\left[\left(4\frac{GM}{rc^2} - \frac{v^2}{c^2}\right)\vec{r} + 4\frac{(\vec{r}\cdot\vec{v})\,\vec{v}}{c^2}\right] \tag{2}$$

The gravitational acceleration is according to GR:

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = -\frac{GM}{r^2}\left(1 - 4\frac{GM}{rc^2} + \frac{v^2}{c^2}\right)\hat{r} + \frac{4GMv^2\,(\hat{r}\cdot\hat{v})}{r^2c^2}\cdot\hat{v} \tag{3}$$

Where:

$\vec{r}$	= distance vector sun - planet	$[m]$
r	= $\lvert\vec{r}\rvert$, magnitude of $\vec{r}$	$[m]$
$\hat{r}$	= unity vector parallel to $\vec{r}$	
$\vec{v}$	= velocity of planet in solar frame	$[m/s]$
v	= $\lvert\vec{v}\rvert$, magnitude of $\vec{v}$	$[m/s]$
$\hat{v}$	= unity vector parallel to $\vec{v}$	
c	= speed of light	$[m/s]$
G	= Gravitational constant	$[m^3\cdot kg^{-1}\cdot s^{-2}]$
M	= solar mass	$[kg]$

3 The perihelion advance predicted by GR

3.1 Derivation of a planet's perihelion advance

In the following will we use the following symbols:

Semi major axis of the orbit	$\boldsymbol{a}$ $[m]$
Eccentricity of the orbit	$\boldsymbol{e}$
Mass of Sun	$\boldsymbol{M}\,[kg]$
Mass of planet	$\boldsymbol{m}$ $[kg]$
Speed of light	$\boldsymbol{c}$ $[m/s]$
Gravitational constant	$\boldsymbol{G}$ $[m^3\cdot kg^{-1}\cdot s^{-2}]$
Angular momentum of planet	$\boldsymbol{L}$ $[kg\cdot m^2/s]$
Perihelion advance per orbit	$\boldsymbol{\delta\phi}$ $[rad/orbit]$
Perihelion advance per century	$\boldsymbol{\delta\phi_c}$ $[arcsec/century]$

According to [2], equation (15.26) page 197, the perihelion advance per orbit is $\epsilon = \dfrac{3m^2}{h^2}$ in natural units.

Converting to SI units:

$$\begin{aligned}
(\epsilon)_{natural\ units} &= \left(\frac{\delta\phi}{2\pi}\right)_{SI\ units} \\
(m)_{natural\ units} &= \left(\frac{GM}{c^2}\right)_{SI\ units} \\
(h)_{natural\ units} &= \left(\frac{L}{mc}\right)_{SI\ units} \\
\left(\frac{3m^2}{h^2}\right)_{natural\ units} &= \left(\frac{3M^2m^2G^2}{L^2c^2}\right)_{SI\ units}
\end{aligned}$$

The perihelion advance $\delta\phi$ is:

$$\delta\phi = \frac{6\pi M^2 m^2 G^2}{L^2 c^2} \tag{4}$$

In the solar frame and the orbital plane:

$\boldsymbol{r}$ = the distance from the Sun to the planet
$\boldsymbol{\theta}$ = the angle between the major axis and line Sun-planet
$\vec{\boldsymbol{v}}$ = the velocity of the planet

We have:

$$r(\theta) = \frac{a\left(1-e^2\right)}{(1+e\cos\theta)} \tag{5}$$

$$|\vec{v}(\theta)| = \sqrt{G(M+m)\left(\frac{2}{r(\theta)} - \frac{1}{a}\right)} \tag{6}$$

The angular momentum $L = mr^2\dot{\theta}$ is constant (independent of θ).

At perihelion: $r = a(1-e)$, $|\vec{v}| = \sqrt{G(M+m)\cdot\dfrac{1+e}{a(1-e)}}$, and $\dot{\theta} = \dfrac{|\vec{v}|}{r}$

So L is:

$$L = m \cdot a(1-e) \cdot |\vec{v}| = m\sqrt{G(M+m)\,a\left(1-e^2\right)} \tag{7}$$

Combining (4) and (7) yields the perihelion advance in radians per orbit:

$$\delta\phi = \frac{6\pi (GM)^2}{G(M+m)\,a\left(1-e^2\right)c^2} \tag{8}$$

The average perihelion advance in arc-seconds per century is:

$$\delta\phi_c = \delta\phi \cdot \frac{100 \cdot y}{P} \cdot \frac{180 \cdot 60 \cdot 60}{\pi} \tag{9}$$

where y is a sidereal year = 365.25636 days and P is the period of the planet in days.

3.2 Calculation of GR's predictions for the perihelion advance of the eight planets

Table 2 shows the relevant data from [3].

Planet	**Mercury**	**Venus**	**Earth**	**Mars**	**Jupiter**	**Saturn**	**Uranus**	**Neptun**
$\boldsymbol{m}$ $[\boldsymbol{kg}]$	3.301132E23	4.867453E24	6.045809E24	6.417094E24	1.898575E27	5.684754E26	8.682160E25	1.024338E26
$\boldsymbol{a}$ $[\mathbf{10^{10}}\boldsymbol{m}]$	5.7909082898	10.820860089	14.959802296	22.793918518	77.829836219	142.93940698	287.50386090	450.44497616
$\boldsymbol{e}$	0.2056317526	0.0067719164	0.0167086342	0.0934006477	0.0484979255	0.0555481426	0.0463812221	0.0094557470

Table 2: Planet data at Epoch J2000

Gravitational constant	$\mathbf{G}$	$= 6.67430 \cdot 10^{-11}$	$m^3 \cdot kg^{-1} \cdot s^{-2}$
Solar mass	$\mathbf{M}$	$= 1.98847 \cdot 10^{30}$	kg
Speed of light	$\mathbf{c}$	$= 299792458$	m/s

Inserting these data in (8) and (9) yields:

Planet	$\boldsymbol{\delta\phi}$ **[radians/orbit]**	$\boldsymbol{\delta\phi_c}$ **[arcsecs/century]**
Mercury	$5.018815 \cdot 10^{-7}$	42.982717
Venus	$2.572418 \cdot 10^{-7}$	8.624984
Earth	$1.861138 \cdot 10^{-7}$	3.838873
Mars	$1.231883 \cdot 10^{-7}$	1.350975
Jupiter	$3.581349 \cdot 10^{-8}$	0.062276
Saturn	$1.952767 \cdot 10^{-8}$	0.013674
Uranus	$9.701907 \cdot 10^{-9}$	0.002382
Neptune	$6.179583 \cdot 10^{-9}$	0.000774

Table 3: Perihelion advcance of the planets

4 Results of the simulation

Planet	Predicted by GR $\delta\phi_c$ ["/century]	Simulated values $\delta\phi_c$ ["/century]	Predicted-Simulated/ Predicted
Mercury	42.982717	42.979	8.65E-5
Venus	8.624984	8.624	1.14E-4
Earth	3.838873	3.838	2.27E-4
Mars	1.350975	1.351	-1.85E-5
Jupiter	0.062276	0.0620	4.43E-3
Saturn	0.013674	0.0136	5.41E-3
Uranus	0.002382	0.00237	5.04E-3
Neptune	0.000774	0.00077	5.17E-3

Table 4: Perihelion advance of the planets

The very good accordance between the simulated values and the predicted values indicates that equation (3) is a very good approximation of the gravitational acceleration predicted by GR.

References

[1] Barman Shahid-Saless1 and Donald K. Yeomans:
RELATIVISTIC EFFECTS ON THE MOTION OF ASTEROIDS AND COMETS

[2] Ray d'Inverno: Introducing Einstein's Relativity
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[3] J.L. Simon & al:
Numerical expressions for precession formulae and mean elements for the Moon and planets