

The rate of clocks in GPS orbit compared to clocks on the geoid

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The Schwarzschild metric is:

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \frac{1}{\left(1 - \frac{2GM}{c^2 r}\right)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (1)$$

where:

- τ is the proper time
- t is the Schwarzschild temporal coordinate
- r is the Schwarzschild radial coordinate
- θ is the colatitude (angle from north)
- φ is the longitude
- G is the gravitational constant
- M is the mass of the Earth in kilograms
- c is the speed of light in vacuum

If we assume that the trajectory of the clock is a circle in the equatorial plane, and its speed as measured in the Schwarzschild frame of reference is v , then we can set $dr = 0$, $d\theta = 0$, $\theta = \frac{\pi}{2}$ and $r d\varphi = v dt$

Thus:

$$d\tau^2 = \left(1 - \frac{2GM}{c^2 r} - \frac{v^2}{c^2}\right) dt^2 \quad (2)$$

or:

$$d\tau = \sqrt{1 - \frac{2GM}{c^2 r} - \frac{v^2}{c^2}} dt \quad (3)$$

A first order approximation is:

$$d\tau = \left(1 - \frac{GM}{c^2 r} - \frac{v^2}{2c^2}\right) dt \quad (4)$$

So the rate of a clock measured in Schwarzschild coordinate time is:

$$f = \frac{d\tau}{dt} = \left(1 - \frac{GM}{c^2 r} - \frac{v^2}{2c^2}\right) \quad (5)$$

For a clock on the ground we get:

$$f_1 = \left(1 - \frac{GM}{c^2 r_1} - \frac{v_1^2}{2c^2}\right) \quad (6)$$

and for the satellite clock:

$$f_2 = \left(1 - \frac{GM}{c^2 r_2} - \frac{v_2^2}{2c^2}\right) \quad (7)$$

where:

- r_1 is the radius of the Earth
- r_2 is the radius of the orbiting clock's orbit
- v_1 is the speed of the Earth clock in the Schwarzschild (ECI) frame
- v_2 is the speed of the orbiting clock in the Schwarzschild frame

The rate of a clock in circular orbit compared to a clock at the surface of the Earth is to a first order approximation:

$$\frac{f_2}{f_1} = \frac{1 - \frac{GM}{c^2 r_2} - \frac{v_2^2}{2c^2}}{1 - \frac{GM}{c^2 r_1} - \frac{v_1^2}{2c^2}} \simeq 1 + \frac{GM}{c^2} \left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{v_1^2 - v_2^2}{2c^2} \quad (8)$$

And the relative rate difference will be:

$$\frac{f_2 - f_1}{f_1} \simeq \frac{GM}{c^2} \left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{v_1^2 - v_2^2}{2c^2} \quad (9)$$

We use the following data:

- Gravitational constant $G = 6.67384 \cdot 10^{-11} \text{ Nm}^2/\text{kg}^2$
- Mass of the Earth $M = 5.97219 \cdot 10^{24} \text{ kg}$
- Speed of light in vacuum $c = 299792458 \text{ m/s}$
- Sidereal day = 86164 s
- Orbital period of GPS satellite = half sidereal day = 43082 s
- Radius of the Earth $r_1 = 6.378 \cdot 10^6 \text{ m}$
- Radius of GPS orbit $r_2 = 26.56 \cdot 10^6 \text{ m}$
- Speed of ground clock $v_1 = \frac{2\pi r_1}{86164s} = 465.09 \text{ m/s}$
- Speed of satellite $v_2 = \frac{2\pi r_2}{43082s} = 3873.57 \text{ m/s}$

Inserting these numbers, we find that what we loosely could call the "time dilation factor" due to gravitation is:

$$\frac{GM}{c^2} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) = 5.2867 \cdot 10^{-10} \quad (10)$$

which is equivalent to 45.55 μs per sidereal day.

The "time dilation factor" due to the velocities will be:

$$\frac{v_1^2 - v_2^2}{2c^2} = -8.2271 \cdot 10^{-11} \quad (11)$$

which is equivalent to $-7.09 \mu s$ per sidereal day.

The relative rate difference will be:

$$\frac{f_2 - f_1}{f_1} = \frac{GM}{c^2} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) + \frac{v_1^2 - v_2^2}{2c^2} = 4.4640 \cdot 10^{-10} \quad (12)$$

which is equivalent to 38.46 μs per sidereal day.

The correct number for the relative rate difference should be $4.4647 \cdot 10^{-10}$.

The discrepancy is mainly due to the oblateness of the Earth, which I have not considered.

For a more rigorous derivation see:

[Neil Ashby: Relativity in the Global Positioning System ↗](#)