The rate of clocks in GPS orbit compared to clocks on the geoid

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The Schwarzschild metric is:

$$c^2\mathrm{d}\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2\mathrm{d}t^2 - \frac{1}{\left(1 - \frac{2GM}{c^2 r}\right)}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) \tag{1}$$

where:

$\boldsymbol{\tau}$	is the proper time
$\boldsymbol{t}$	is the Schwarzschild temporal coordinate
$\boldsymbol{r}$	is the Schwarzschild radial coordinate
$\boldsymbol{\theta}$	is the colatitude (angle from north)
$\boldsymbol{\varphi}$	is the longitude
$\boldsymbol{G}$	is the gravitational constant
$\boldsymbol{M}$	is the mass of the Earth in kilograms
$\boldsymbol{c}$	is the speed of light in vacuum

If we assume that the trajectory of the clock is a circle in the equatorial plane, and its speed as measured in the Schwarzschild frame of reference is v, then we can set
$\mathrm{d}r = 0$, $\mathrm{d}\theta = 0$, $\theta = \frac{\pi}{2}$ and $r\,\mathrm{d}\varphi = v\,\mathrm{d}t$

Thus:

$$\mathrm{d}\tau^2 = \left(1 - \frac{2GM}{c^2 r} - \frac{v^2}{c^2}\right)\mathrm{d}t^2 \tag{2}$$

or:

$$\mathrm{d}\tau = \sqrt{1 - \frac{2GM}{c^2 r} - \frac{v^2}{c^2}}\,\mathrm{d}t \tag{3}$$

A first order approximation is:

$$\mathrm{d}\tau \simeq \left(1 - \frac{GM}{c^2 r} - \frac{v^2}{2c^2}\right)\mathrm{d}t \tag{4}$$

So the rate of a clock measured in Schwarzschild coordinate time is:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} \simeq \left(1 - \frac{GM}{c^2 r} - \frac{v^2}{2c^2}\right) \tag{5}$$

For a clock on the ground we get:

$$\frac{\mathrm{d}\tau_1}{\mathrm{d}t} \simeq \left(1 - \frac{GM}{c^2 r_1} - \frac{v_1^2}{2c^2}\right) \tag{6}$$

and for the satellite clock:

$$\frac{\mathrm{d}\tau_2}{\mathrm{d}t} \simeq \left(1 - \frac{GM}{c^2 r_2} - \frac{v_2^2}{2c^2}\right) \tag{7}$$

where:

- $\boldsymbol{r_1}$ is the radius of the Earth
- $\boldsymbol{r_2}$ is the radius of the orbiting clock's orbit
- $\boldsymbol{v_1}$ is the speed of the Earth clock in the Schwarzschild (ECI) frame
- $\boldsymbol{v_2}$ is the speed of the orbiting clock in the Schwarzschild frame

The rate of a clock in circular orbit compared to a clock at the surface of the Earth is to a first order approximation:

$$\frac{\mathrm{d}\tau_2}{\mathrm{d}\tau_1} = \frac{1 - \dfrac{GM}{c^2 r_2} - \dfrac{v_2^2}{2c^2}}{1 - \dfrac{GM}{c^2 r_1} - \dfrac{v_1^2}{2c^2}} \simeq 1 + \frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{v_1^2 - v_2^2}{2c^2} \tag{8}$$

And the relative rate difference will be:

$$\frac{\mathrm{d}\tau_2}{\mathrm{d}\tau_1} - 1 \simeq \frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{v_1^2 - v_2^2}{2c^2} \tag{9}$$

We use the following data:

Gravitational constant $G = 6.67384 \cdot 10^{-11}\ Nm^2/kg^2$
Mass of the Earth $M = 5.97219 \cdot 10^{24}\ kg$
Speed of light in vacuum $c = 299792458\ m/s$
Sidereal day $= 86164\ s$
Orbital period of GPS satellite = half sidereal day $= 43082\ s$
Radius of the Earth $r_1 = 6.378 \cdot 10^6$ m
Radius of GPS orbit $r_2 = 26.56 \cdot 10^6$ m
Speed of ground clock $v_1 = \frac{2\pi r_1}{86164s} = 465.09\ m/s$
Speed of satellite $v_2 = \frac{2\pi r_2}{43082s} = 3873.57\ m/s$

Inserting these numbers, we find that what we loosely could call the "time dilation factor" due to gravitation is:

$$\frac{GM}{c^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) = 5.2867 \cdot 10^{-10} \tag{10}$$

which is equivalent to 45.55 μs per sidereal day.

The "time dilation factor" due to the velocities will be:

$$\frac{v_1^2 - v_2^2}{2c^2} = -8.2271 \cdot 10^{-11} \tag{11}$$

which is equivalent to -7.09 μs per sidereal day.

The relative rate difference will be:

$$\frac{\mathrm{d}\tau_2}{\mathrm{d}\tau_1} - 1 \simeq \frac{GM}{c^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) + \frac{v_1^2 - v_2^2}{2c^2} = 4.4640 \cdot 10^{-10} \tag{12}$$

which is equivalent to 38.46 μs per sidereal day.

The correct number for the relative rate difference should be $4.4647 \cdot 10^{-10}$.
The discrepancy is mainly due to the oblateness of the Earth, which I have not considered.

For a more rigorous derivation see:
Neil Ashby: Relativity in the Global Positioning System