Pat Dolan's question

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April 29, 2020

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On April 6, 2020, Pat Dolan posted the following question in the Usenet group sci.physics.relativity:

> *"A distant observer is traveling at .9c relative to the solar system along the line that is collinear with the sun's axis of rotation. As the clockwork solar system spins beneath the observer he/she observes the earth following an elliptical path around the sun. How much time does the observer measure on his/her wristwatch for the earth to complete 2pi radians on the aforementioned path?"*

Answer when the curvature of space-time is ignored

When gravitation is ignored, we can apply the Special Theory Of Relativity (SR).

The question can be interpreted in at least three different ways, all of them trivially simple.

A "*year*" is defined as the time it takes for Earth to orbit the Sun once, as measured on the Earth. So $year = 31558149.76$ seconds.

Interpretation #1:

The orbiting Earth is a clock which is visually observed by the observer. This clock is approaching the observer with the speed 0.9c.

The observer will the measure one orbit of the Earth to last a time τ on his wristwatch:

$$\tau = \sqrt{\frac{\left(1-\frac{v}{c}\right)}{\left(1+\frac{v}{c}\right)}}\, year = 0.229416\, year \tag{1}$$

Interpretation #2:

The observer is stationary in a frame of reference. The orbiting Earth is a clock which is moving at 0.9c in the observer's rest frame.

The duration of one Earth orbit measured in the observer's frame will then be:

$$t = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\,year = 2.294157\,year \tag{2}$$

Interpretation #3:

The observer is moving at 0.9c in the Solar frame. The proper time of the observer's wristwatch will advance a time τ while one year passes as measured in the solar frame.

$$\tau = \sqrt{1-\frac{v^2}{c^2}}\,year = 0.43588989\,year \tag{3}$$

Answer when the curvature of space-time is considered

A spaceship is moving at the speed v towards the Sun. The proper time of the spaceship's clock will advance a time τ while one year passes as measured in the solar frame.

In this case we must use The General Theory of Relativity (GR), which makes the problem much more interesting. We will use Schwarzschild coordinates as coordinate system in the solar frame.

The Schwarzschild metric is:

$$c^2 \mathrm{d}\tau^2 = \left(1-\frac{2GM}{c^2 r}\right)c^2\mathrm{d}t^2 - \frac{1}{\left(1-\frac{2GM}{c^2 r}\right)}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) \tag{4}$$

where:

$\boldsymbol{\tau}$ is the proper time
$\boldsymbol{t}$ is the Schwarzschild temporal coordinate
$\boldsymbol{r}$ is the Schwarzschild radial coordinate
$\boldsymbol{\theta}$ is the colatitude (angle from north)
$\boldsymbol{\varphi}$ is the longitude
$\boldsymbol{G}$ is the gravitational constant
$\boldsymbol{M}$ is the mass of the Sun
$\boldsymbol{c}$ is the speed of light in vacuum

We will first calculate the Schwarzschild time it takes for the Earth to orbit the Sun once. A clock is at the North pole of the Earth. If we assume that the trajectory of the clock is a circle with radius r = 1 AU in the ecliptic plane, and its speed as measured in the

Schwarzschild frame of reference is v_e, then we can set $\mathrm{d}r = 0$, $\mathrm{d}\theta = 0$, $\theta = \dfrac{\pi}{2}$ and $r\mathrm{d}\varphi = v_e \mathrm{d}t$.
Thus:

$$\mathrm{d}\tau_c^2 = \left(1 - \frac{2GM}{c^2 r} - \frac{2GM_e}{c^2 r_e} - \frac{v_e^2}{c^2}\right) \mathrm{d}t^2 \tag{5}$$

where M_e is the mass of the Earth and r_e is the polar radius of the Earth.

$$\mathrm{d}t = \left(1 - \frac{2GM}{c^2 r} - \frac{2GM_e}{c^2 r_e} - \frac{v_e^2}{c^2}\right)^{-\frac{1}{2}} \mathrm{d}\tau_c \tag{6}$$

We define *year* as one sidereal year = 31558149.76 seconds.

The clock will during one orbit around the Sun advance $\tau_c = 1\ year$, so the one orbit of the Earth will in Schwarzschild time be:

$$T = \left(1 - \frac{2GM}{c^2 r} - \frac{2GM_e}{c^2 r_e} - \frac{v_e^2}{c^2}\right)^{-\frac{1}{2}} year = \left(1 + 1.55 \cdot 10^{-8}\right) year \tag{7}$$

If we assume that the trajectory of the the spaceship is a straight line along the axis through the Sun and perpendicular to the ecliptic plane and its speed as measured in the Schwarzschild frame of reference is v, then we can set $\mathrm{d}\theta = 0$, $\theta = 0$, and $\mathrm{d}\varphi = 0$.

Thus:

$$\mathrm{d}\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) \mathrm{d}t^2 - \frac{1}{c^2\left(1 - \frac{2GM}{c^2 r}\right)} \mathrm{d}r^2 \tag{8}$$

We set $r_s = \frac{2GM}{c^2} = 2953.25$ m and if we assume r is in the order of 1 AU, we have $\frac{r_s}{r} \ll 1$ and we can use the approximation $\dfrac{1}{1-x} \approx 1 + x$ when $x \ll 1$.

$$\mathrm{d}\tau^2 \approx \left(1 - \frac{r_s}{r}\right) \mathrm{d}t^2 - \left(1 + \frac{r_s}{r}\right) \frac{\mathrm{d}r^2}{c^2} \tag{9}$$

Since the spaceship is moving at the speed v towards the Sun, we have $\mathrm{d}r = -v \cdot \mathrm{d}t$

$$\mathrm{d}\tau^2 \approx \left(1 - \frac{r_s}{r}\right) \mathrm{d}t^2 - \left(1 + \frac{r_s}{r}\right) \cdot \frac{v^2}{c^2} \mathrm{d}t^2 = \left(1 - \frac{v^2}{c^2}\right) \cdot \left(1 - \frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}} \cdot \frac{r_s}{r}\right) \mathrm{d}t^2 \tag{10}$$

$$\mathrm{d}\tau \approx \sqrt{1 - \frac{v^2}{c^2}} \cdot \sqrt{(1 - \frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}} \cdot \frac{r_s}{r}} \cdot \mathrm{d}t \tag{11}$$

Now we use the approximation: $\sqrt{1-x} \approx \left(1 - \frac{x}{2}\right)$ when $x \ll 1$

$$\mathrm{d}\tau \approx \sqrt{1 - \frac{v^2}{c^2}} \cdot \left(1 - \frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}} \cdot \frac{r_s}{2r}\right) \mathrm{d}t \tag{12}$$

Now we set $r = -vt$

$$d\tau \approx \sqrt{1-\frac{v^2}{c^2}}\,dt + \left(\frac{1+\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\frac{r_s}{2v}\right)\frac{dt}{t} \tag{13}$$

During one orbit of the Earth, the spaceship moves from $r = r_0 + vT$ to r_0 where r_0 is 1 AU so that the approximation in equation (12) is valid. That means that to find τ we must integrate equation (13) from $t = -\frac{r_0}{v} - T$ to $t = -\frac{r_0}{v}$.

$$\tau = \sqrt{1-\frac{v^2}{c^2}}\cdot\int_{-\frac{r_0}{v}-T}^{-\frac{r_0}{v}} dt + \frac{r_s\left(1+\frac{v^2}{c^2}\right)}{2v\sqrt{1-\frac{v^2}{c^2}}}\cdot\int_{-\frac{r_0}{v}-T}^{-\frac{r_0}{v}}\frac{dt}{t} \tag{14}$$

$$\tau = \sqrt{1-\frac{v^2}{c^2}}\cdot T + \frac{r_s\left(1+\frac{v^2}{c^2}\right)}{2v\sqrt{1-\frac{v^2}{c^2}}}\cdot\ln\left(\frac{r_0}{r_0+vT}\right) \tag{15}$$

With $v = 0.9c$ we get:

$$\tau = 0.4358898944\cdot\left(1+1.550\cdot 10^{-8}\right) year - 4.095\cdot 10^{-8} year = 0.43588986\ year \tag{16}$$

This is very close to the same value as SR gave, the difference is $3\cdot 10^{-8}$ *year* or ca 1 second. So SR is a good approximation in this case.