Setting synchronized clocks in motion

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1 Introduction

We will consider what the Special Theory of Relativity predicts for the following scenario:

Two clocks, A and B, are stationary in an inertial frame K. The distance between them is L, and the clocks are synchronized according to Einstein's procedure. At some time, the clocks are instantly set in motion to the speed v in K so that they become stationary in an inertial frame K' which is moving at the speed v relative to K.

By "instantly set in motion" we mean a brief, very high acceleration a for a very short time Δt such that

$$\lim_{\Delta t \to 0} (a\Delta t) = v \tag{1}$$

Will the clocks be synchronous in K', and if not, how much out of sync will they be?

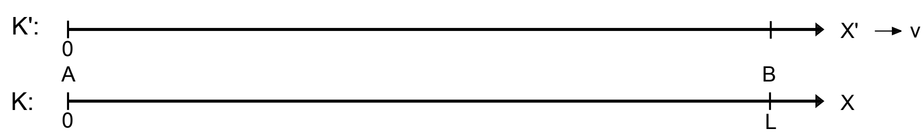

Figure 1: *The frames of reference*

2 Calculation of how much the clocks are out of sync in K'

As the well informed reader will know, the answer depends on how the clocks are set in motion. We will consider a few alternatives.

In all the cases, we have:

- Clock A is set in motion at the event $t = t' = 0$ when the origins of K and K' are aligned. That means that after clock A is set in motion, it will show the coordinate time of K'.
- x'_1 is the position in K' where clock B will be when it is stationary in K', t_1 is what clock B will show at the event when it is set in motion, and t'_1 is the coordinate time of K' at the same event. That is, t'_1 is what clock B should have shown to be synchronous to clock A, and clock B will thus be $(t_1 - t'_1)$ ahead of clock A after it is set in motion.

2.1 The clocks are set in motion simultaneously in K

The clocks are instantly set in motion when they both show 0.

We have the following events of interest:

1. Event E_0: Clock A is set in motion, showing 0.
2. Event E_1: Clock B is set in motion, showing 0.

The coordinates of the events in K are:

$$E_0: \quad x_0 = 0 \quad t_0 = 0 \tag{2}$$
$$E_1: \quad x_1 = L \quad t_1 = 0 \tag{3}$$

The coordinates of the events in K' are:

$$E_0: \qquad x'_0 = 0 \qquad t'_0 = 0 \tag{4}$$
$$E_1: \qquad x'_1 = \frac{x_1 - vt_1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{L}{\sqrt{1-\frac{v^2}{c^2}}} \qquad t'_1 = \frac{t_1 - \frac{v}{c^2}x_1}{\sqrt{1-\frac{v^2}{c^2}}} = -\frac{vL}{c^2\sqrt{1-\frac{v^2}{c^2}}} \tag{5}$$

So the conclusion is that the clocks will not be synchronous in K', clock B will be $\frac{vL}{c^2\sqrt{1-\frac{v^2}{c^2}}}$ ahead of clock A, and the distance between the clocks is lengthened to $\frac{L}{\sqrt{1-\frac{v^2}{c^2}}}$.

2.2 The clocks are set in motion simultaneously in K'

The clocks are, simultaneously in K', instantly set in motion when clock A shows 0.

We have the following events of interest:

1. Event E_0: Clock A is set in motion, showing 0.
2. Event E_1: Clock B is set in motion, showing t_1.

The coordinates of the events in K are:

$$E_0: \qquad x_0 = 0 \qquad t_0 = 0 \tag{6}$$
$$E_1: \qquad x_1 = L \qquad t_1 = \frac{t'_1 + \frac{v}{c^2}\ x'_1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\frac{v}{c^2}\ x'_1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{7}$$

The coordinates of the events in K' are:

$$E_0: \qquad x'_0 = 0 \qquad t'_0 = 0 \tag{8}$$
$$E_1: \qquad x'_1 = \frac{x_1 - vt_1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{L - vt_1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad t'_1 = 0 \tag{9}$$

Solving the equations:

$$t_1 = \frac{\frac{v}{c^2}\ x'_1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{10}$$
$$x'_1 = \frac{L - vt_1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{11}$$

yields:

$$t_1 = \frac{vL}{c^2} \qquad x'_1 = \sqrt{1 - \frac{v^2}{c^2}}\ L \tag{12}$$

So the conclusion is that the clocks will not be synchronous in K', clock B will be $\frac{vL}{c^2}$ ahead of clock A, and the distance between the clocks is shortened to $L\sqrt{1 - \frac{v^2}{c^2}}$.

2.3 The clocks are set in motion such that the distance between them is retained

We assume that the clocks are attached to a rigid rod, and the rod with the clocks is set in motion such that there is no stress in the rod during the acceleration.

We have the following events of interest:

1. Event E_0: Clock A is set in motion, showing 0.
2. Event E_1: Clock B is set in motion, showing t_1.

The coordinates of the events in K are:

$$E_0 : \qquad x_0 = 0 \qquad t_0 = 0 \tag{13}$$

$$E_1 : \qquad x_1 = L \qquad t_1 = \frac{t'_1 + \frac{vL}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{14}$$

The coordinates of the events in K' are:

$$E_0 : \qquad x'_0 = 0 \qquad t'_0 = 0 \tag{15}$$

$$E_1 : \qquad x'_1 = L \qquad t'_1 = \frac{t_1 - \frac{vL}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{16}$$

Solving the equations:

$$t_1 = \frac{t'_1 + \frac{vL}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{17}$$

$$t'_1 = \frac{t_1 - \frac{vL}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{18}$$

yields:

$$t_1 = \left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right)\frac{L}{v} \qquad t'_1 = -\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right)\frac{L}{v} \tag{19}$$

The conclusion is that the clocks will not be synchronous in K', clock B will be $2\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right)\frac{L}{v}$ ahead of clock A when the clocks are set in motion so that the distance between them is retained.

3 Conclusion

The clocks will not be synchronous after they are set in motion, the leading clock will be ahead of the trailing clock by an amount which will depend on how they are set in motion.

If the clocks are set in motion simultaneously in K, then clock B will be $\frac{vL}{c^2\sqrt{1-\frac{v^2}{c^2}}}$ ahead of clock A, and the distance between the clocks is lengthened to $\frac{L}{\sqrt{1-\frac{v^2}{c^2}}}$.

If the clocks are set in motion simultaneously in K', then clock B will be $\frac{vL}{c^2}$ ahead of clock A, and the distance between the clocks is shortened to $L\sqrt{1-\frac{v^2}{c^2}}$.

If the clocks are set in motion such that the distance between them is retained, then clock B will be $2\left(1-\sqrt{1-\frac{v^2}{c^2}}\right)\frac{L}{v}$ ahead of clock A.

Note that to a first order approximation in $\frac{v}{c}$, clock B will be $\frac{vL}{c^2}$ ahead of clock A in all three cases.

APPENDIX

What happens during the acceleration?

If we have two clocks, A and B, in a frame of reference with an acceleration a, and the acceleration is aligned along a line through A and B towards B, then the rate of clock B compared to clock A will to a first order approximation be: $\left(1+\frac{aL}{c^2}\right)$ where L is the distance between the clocks. (A clock 'higher up' in a pseudo gravitational field runs faster, see Pound & Rebka)

If this acceleration lasts for a time Δt, then clock B will be a time ΔT ahead of clock A:

$$\Delta T \approx \left(1+\frac{aL}{c^2}\right)\Delta t - \Delta t = \frac{a\Delta t\ L}{c^2} \tag{20}$$

If the acceleration is very brief and high, such that:

$$\lim_{\Delta t\to 0}(a\Delta t) = v \tag{21}$$

then:

$$\lim_{\Delta t\to 0}\Delta T \approx \lim_{\Delta t\to 0}\frac{a\Delta t\ L}{c^2} = \frac{vL}{c^2} \tag{22}$$

Which is equal to the approximated value we found above.