The rate of clocks in circular orbit compared to clocks on the geoid

Paul B. Andersen

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1 What is *time*?

1.1 What is proper time?

In physics, *time* must be measurable. *Proper time* is what we measure with clocks. There is no alternative to this definition. A *proper clock* must be independent of environmental parameters which may vary where the clock is used. These parameters can be temperature, air pressure, acceleration, etc. Any clock may serve as a proper clock if the resolution and precision is adequate for the purpose at hand, but the best clocks we have are atomic clocks based on the frequency of the photon associated with a hyperfine transition. Only atomic clocks will be adequate for our purpose.

SI has defined the time unit *second* like this:

> ***The duration of 9,192,631,770 periods of the radiation corresponding to the two hyperfine levels of the ground state of the caesium-133 atom.***

In the following we will call clocks which use this definition of second *an SI-clock.* Note that SI-clocks per definition always run at the rate 1 second per second.

1.2 What is coordinate time?

Coordinate time is the temporal coordinate in a coordinate system. The coordinate time of a coordinate system can be defined in several different ways, but we will use two different coordinate times, the Schwarzschild coordinate time and the Coordinated Universal Time (UTC).

1.2.1 Schwarzschild coordinate time

The Schwarzschild coordinate system is a non rotating spherical coordinate system. The coordinate time of any spatial point is at any time the same, and the rate of the time is such that an SI-clock at infinity would stay in sync (have the same rate).

Note that all clocks showing Schwarzschild coordinate time are synchronous in the non rotating Schwarzschild coordinate system, they are *not* synchronous in the rotating Earth-fixed coordinate system.

1.2.2 Coordinated Universal Time (UTC)

The coordinate system is still the Schwarzschild coordinate system, but with a different temporal coordinate. We will call the UTC coordinate t_{utc}. The coordinate time of any spatial point is at any time the same, and the rate is such that a stationary SI-clock on Earth's geoid will stay in sync with UTC. Note that this SI-clock is moving in the Schwarzschild coordinate system.

Note that all clocks showing UTC coordinate time are synchronous in the non rotating Schwarzschild coordinate system, they are *not* synchronous in the rotating Earth-fixed coordinate system.

The only difference between Schwarzschild coordinate time and UTC is the rate. One might think that we could use equation (5) below to find this rate difference $\frac{\mathrm{d}t_{utc}}{\mathrm{d}t}$, but due to the fact that the Earth is not a non rotating perfect sphere, but a rotating ellipsoid, it is not quite that simple. A point on equator and a point on the North pole are both on the geoid. If we set r equal to the equatorial radius of the Earth and v equal to the velocity of a point on equator, we find $\frac{\mathrm{d}t_{utc}}{\mathrm{d}t} = (1 - 6.96552 \cdot 10^{-10})$, but if we set r equal to the polar radius of the Earth, and $v = 0$ we get $\frac{\mathrm{d}t_{utc}}{\mathrm{d}t} = (1 - 6.97688 \cdot 10^{-10})$. Since we know that the rates of the clocks on the geoid are equal, this shows that equation (5) can not be used.

This problem is considered by Neil Ashby in [1], see equation (18) page 11.
According to Ashby: $\frac{\mathrm{d}t_{utc}}{\mathrm{d}t} = (1 - 6.96927{\cdot}10^{-10})$.

We define: $\delta_{utc} = 6.96927{\cdot}10^{-10}$

In the following we will use:

$$\frac{\mathrm{d}t_{utc}}{\mathrm{d}t} = (1 - \delta_{utc}) \tag{1}$$

2 Rate of clocks in circular orbit

2.1 Rate of clocks in circular orbit compared to Schwarzschild time

The Schwarzschild metric is:

$$c^2\mathrm{d}\tau^2 = \left(1 - \frac{2GM}{c^2r}\right)c^2\mathrm{d}t^2 - \frac{1}{\left(1 - \frac{2GM}{c^2r}\right)}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) \tag{2}$$

where:

$\boldsymbol{\tau}$	is the proper time
$\boldsymbol{t}$	is the Schwarzschild temporal coordinate
$\boldsymbol{r}$	is the Schwarzschild radial coordinate
$\boldsymbol{\theta}$	is the colatitude (angle from north)
$\boldsymbol{\varphi}$	is the longitude
$\boldsymbol{G}$	is the gravitational constant
$\boldsymbol{M}$	is the mass of the Earth
$\boldsymbol{c}$	is the speed of light in vacuum

If we assume that the orbit of the clock is circular in the equatorial plane, we can set $\mathrm{d}r = 0$, $\mathrm{d}\theta = 0$, $\theta = \frac{\pi}{2}$ and $r\,\mathrm{d}\varphi = r\,\omega\,\mathrm{d}t = v\,\mathrm{d}t$ where v is the orbital speed of the clock. The orbital speed will however be the same for all circular orbits with the same radius, so the equation below is equally valid for all circular orbits.

$$\mathrm{d}\tau^2 = \left(1 - \frac{2\mu}{c^2r} - \frac{v^2}{c^2}\right)\mathrm{d}t^2 \tag{3}$$

where $\mu = G{\cdot}M$ is the geocentric gravitational constant.

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{1 - \frac{2\mu}{c^2r} - \frac{v^2}{c^2}} \tag{4}$$

A very good approximation is:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} \simeq 1 - \frac{\mu}{c^2r(t)} - \frac{v(t)^2}{2c^2} \tag{5}$$

The difference between (4) and (5) is less than 10^{-25} for all r.

This equation is valid for any circular orbit. The term $\frac{\mu}{c^2r(t)}$ is called the gravitational term, while the term $\frac{v(t)^2}{2c^2}$ is called the kinematic term.

Since the orbit is circular, r and v are constants, and we have:

$$\frac{v^2}{r} = \frac{\mu}{r^2} \quad \Rightarrow \quad \frac{v^2}{2c^2} = \frac{\mu}{2c^2r} \tag{6}$$

The rate of an SI-clock in circular orbit relative to the Schwarzschild coordinate time t is:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = 1 - \frac{1.5\mu}{c^2 r} \tag{7}$$

This is a real rate difference, but note that the SI-clock runs at its proper rate, one second per second. It is the Schwarzschild coordinate time that is fast, and its seconds are shorter than the SI-second.

2.2 Rate of clocks in circular orbit compared to UTC

The rate of SI-clocks in circular orbit relative to UTC is:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t_{utc}} = \frac{1 - \frac{1.5\mu}{c^2 r}}{\frac{\mathrm{d}t_{utc}}{\mathrm{d}t}} = \frac{1 - \frac{1.5\mu}{c^2 r}}{1 - \delta_{utc}} \tag{8}$$

A good approximation is:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t_{utc}} \simeq \left(1 - \frac{1.5\mu}{c^2 r}\right) \cdot (1 + \delta_{utc}) \simeq 1 - \frac{1.5\mu}{c^2 r} + \delta_{utc} \tag{9}$$

The difference between (8) and (9) is less than 10^{-17} for all r.

Let f be the rate of the rate of the clock in circular orbit and f_0 the rate of the UTC:

$$\frac{\Delta f}{f_0} = \frac{f - f_0}{f_0} = \left(\frac{\mathrm{d}\tau}{\mathrm{d}t_{utc}} - 1\right) \simeq \delta_{utc} - \frac{1.5\mu}{c^2 r} \tag{10}$$

3 Rate of clock in the orbits of the GNSS satellites GPS, Galileo and GLONASS

3.1 Common data

Geocentric gravitational constant $\mu = 3.986004418 \cdot 10^{14}\ \frac{m^3}{s^2}$
Speed of light in vacuum $c = 299792458\ \frac{m}{s}$
Sidereal day $= 86164.0905\ s$
Equatorial radius of the Earth $R = 6378137\ m$

3.2 GPS

According to [3] the GPS orbit is circular with period half a sidereal day, $p = 43082.04525\ s$
The radius of the orbit is then $r = \frac{GM \cdot p^2}{4\pi^2} = 26561763\ m$.

Equation (10) gives: $\frac{\Delta f}{f_0} = 4.4647 \cdot 10^{-10}$

3.3 Galileo

According to [4] the Galileo orbit is circular with orbital radius $r = 29600000\ m$.

Equation (10) gives: $\frac{\Delta f}{f_0} = 4.7218 \cdot 10^{-10}$

3.4 GLONASS

According to [5] the GLONASS orbit is circular with altitude $h = 19100000\ m$, the orbital radius is then $r = h + R = 25478137\ m$.

Equation (10) gives: $\frac{\Delta f}{f_0} = 4.3582 \cdot 10^{-10}$

3.5 Geostationary satellite

The orbit of a geostationary satellite is circular with period a sidereal day, $p = 86164.0905\ s$
The radius of the orbit is then $r = \frac{GM \cdot p^2}{4\pi^2} = 42164169\ m$.

Equation (10) gives: $\frac{\Delta f}{f_0} = 5.3915 \cdot 10^{-10}$

REFERENCES

[1] Neil Ashby:
Relativity in the Global Positioning System

[2] Dr. A. Mudraka, Dr. P. De Simonea, Dr. ing. M. Lisi:
Relativistic Corrections in the European GNSS Galileo

[3] U.S Government:
GPS INTERFACE SPECIFICATION DOCUMENT

[4] European GNSS Agency:
Galileo Interface Control Document

[5] Russian Institute of Space Device Engineering:
GLONASS INTERFACE CONTROL DOCUMENT