Aberration and Doppler shift of EM-waves

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1 Introduction

This is 'yet another' calculation of relativistic Doppler shift and aberration, the reader will be able to find the same many places.

We will in the following see how a plane electromagnetic wave in vacuum can be expressed mathematically in two different inertial frames of reference.

In the inertial frame of reference K, the wave is propagating at the speed c in a direction perpendicular to the z-axis and at an angle θ to the x-axis, see fig.1. Another inertial frame of reference K', with the x' and y' axes aligned with the x and y axes of K, is moving with the speed v along the x-axis of K. In this frame of reference the same wave is propagating at the speed c at an angle θ' to the x'-axis, see fig. 2.

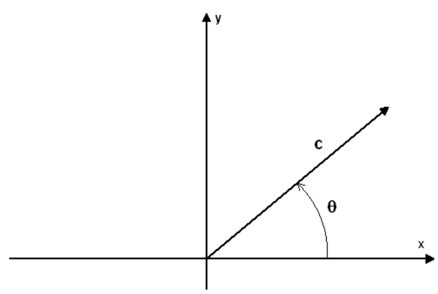

Figure 1: *Frame of reference K*

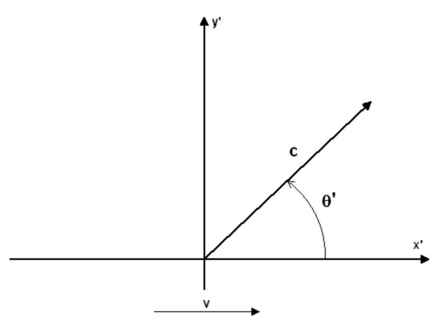

Figure 2: *Frame of reference K'*

2 Transformation of the wave

The electric field in a plane, monochromatic, linearly polarized electromagnetic wave can be written as a function of the coordinates in the frame of reference K:

$$E\left(t,\vec{r}\right) = E_0 \cos\left(\Phi\left(t,\vec{r}\right)\right) \tag{1}$$

$$\Phi\left(t,\vec{r}\right) = \omega t - \vec{k}\cdot\vec{r} \tag{2}$$

where E is the electric field, E_0 is the constant amplitude of the electric field, t is the temporal coordinate, $\vec{r}$ is the position vector, ω is the angular frequency of the wave, and $\vec{k}$ is the wave vector.

The components of $\vec{k}$ and $\vec{r}$ in the frame of reference K are:

$$k_x = \frac{\omega}{c}\cos\theta \qquad k_y = \frac{\omega}{c}\sin\theta \qquad k_z = 0 \tag{3}$$

$$r_x = x \qquad r_y = y \qquad r_z = 0 \tag{4}$$

Thus:

$$\vec{k}\cdot\vec{r} = \frac{\omega}{c}\, x\cos\theta + \frac{\omega}{c}\, y\sin\theta \tag{5}$$

The phase of the wave can be written as a function of the coordinates in the frame of reference K:

$$\Phi\left(t,x,y\right) = \omega t - \frac{\omega}{c}\, x\cos\theta - \frac{\omega}{c}\, y\sin\theta \tag{6}$$

The phase of the same wave can be written as a function of the coordinates in the frame of reference K':

$$\Phi'\left(t',x',y'\right) = \omega' t' - \frac{\omega'}{c}\, x'\cos\theta' - \frac{\omega'}{c}\, y'\sin\theta' \tag{7}$$

Applying the Lorentz transform:

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right),\ x = \gamma\left(x' + vt'\right),\ y = y',\ \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{8}$$

Inserting these in (6) yields:

$$\begin{aligned}\Phi'\left(t',x',y'\right) &= \omega\gamma\left(t' + \frac{vx'}{c^2}\right) - \frac{\omega}{c}\gamma\left(x' + vt'\right)\cos\theta - \frac{\omega}{c}y'\sin\theta & (9)\\ &= \omega\gamma\left(1 - \frac{v}{c}\cos\theta\right)t' - \frac{\omega}{c}\gamma\left(\cos\theta - \frac{v}{c}\right)x' - \frac{\omega}{c}y'\sin\theta & (10)\end{aligned}$$

By comparison with (7), we get the following set of equations:

$$\omega' t' = \omega\gamma\left(1 - \frac{v}{c}\cos\theta\right)t' \tag{11}$$

$$\frac{\omega'}{c}\, x'\cos\theta' = \frac{\omega}{c}\gamma\left(\cos\theta - \frac{v}{c}\right)x' \tag{12}$$

$$\frac{\omega'}{c}\, y'\sin\theta' = \frac{\omega}{c}y'\sin\theta \tag{13}$$

Solving these with respect to ω' and θ' yields:

$$\omega' = \omega\frac{\left(1 - \frac{v}{c}\cos\theta\right)}{\sqrt{1-\frac{v^2}{c^2}}} \tag{14}$$

$$\cos\theta' = \frac{\cos\theta - \frac{v}{c}}{1 - \frac{v}{c}\cos\theta} \tag{15}$$

Alternatively, these can be written;

$$\omega' = \omega \frac{\sqrt{1-\frac{v^2}{c^2}}}{\left(1+\frac{v}{c}\cos\theta'\right)} \tag{16}$$

$$\cos\theta = \frac{\cos\theta' + \frac{v}{c}}{1+\frac{v}{c}\cos\theta'} \tag{17}$$

3 Aberration

Aberration of an EM-wave is the phenomenon that the direction of the propagation velocity of the wave is dependent on the frame of reference.

Equation (15) and (17) give the the aberration:

$$\cos\theta' = \frac{\cos\theta - \frac{v}{c}}{1-\frac{v}{c}\cos\theta} \tag{18}$$

$$\cos\theta = \frac{\cos\theta' + \frac{v}{c}}{1+\frac{v}{c}\cos\theta'} \tag{19}$$

Note that there is no aberration when the wave vector is parallel to the relative velocity between the frames of reference, $\theta' = 0$ when $\theta = 0$, and $\theta' = \pi$ when $\theta = \pi$.

When the wave vector is perpendicular to the velocity of K' in K, $\theta = \frac{\pi}{2}$ then $\theta' = \arccos\left(-\frac{v}{c}\right)$

When the wave vector is perpendicular to the velocity of K in K', $\theta' = \frac{\pi}{2}$ then $\theta = \arccos\left(\frac{v}{c}\right)$

4 Doppler shift

Equation (14) and (16) give the Doppler shift:

$$\omega' = \omega \frac{\left(1-\frac{v}{c}\cos\theta\right)}{\sqrt{1-\frac{v^2}{c^2}}} \tag{20}$$

$$\omega' = \omega \frac{\sqrt{1-\frac{v^2}{c^2}}}{\left(1+\frac{v}{c}\cos\theta'\right)} \tag{21}$$

Longitudinal Doppler shift

When the wave vector is parallel to the velocity of K' in K, and in the same direction, $\theta' = \theta = 0$:

$$\omega' = \omega\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \tag{22}$$

we have a red shift.

When the wave vector is parallel to the velocity of K' in K, and in the opposite direction, $\theta' = \theta = \pi$:

$$\omega' = \omega\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \tag{23}$$

we have a blue shift.

Transverse Doppler shift

When the wave vector is perpendicular to the velocity of K' in K, $\theta = \frac{\pi}{2}$:

$$\omega' = \omega \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{24}$$

we have a blue shift.

When the wave vector is perpendicular to the velocity of K in K', $\theta' = \frac{\pi}{2}$:

$$\omega' = \omega \sqrt{1 - \frac{v^2}{c^2}} \tag{25}$$

we have a red shift.

5 Conclusion

We have shown that if the angular frequency of an electromagnetic wave is ω, and the angle between the wave vector and the x-axis is θ in the frame of reference K, then the angular frequency and the angle between the wave vector and the x'-axis in the frame of reference K', which is moving with the the speed v along the x-axis of K, are respectively $\omega' = \omega \frac{\sqrt{1 - \frac{v^2}{c^2}}}{\left(1 + \frac{v}{c}\cos\theta'\right)}$ and $\cos\theta' = \frac{\cos\theta - \frac{v}{c}}{1 - \frac{v}{c}\cos\theta}$.

It is utterly irrelevant if the source of the wave is stationary in any of the two frames, Doppler shift and aberration are the phenomena that the frequency and direction of the wave are different in two relatively moving frames of reference.

There is but one wave, and the frequency and direction are frame dependent.

It is a strange misconception among some of the less knowledgeable contributors in the newsgroup sci.physics.relativity that Doppler shift and aberration is only between the rest frame of the source and the rest frame of the observer. A reason for this may be that Einstein in his 1905 paper "On the Electrodynamics of moving Bodies" chose these two frames of reference in his treatment of the subject.