FOURTH TEST OF GENERAL RELATIVITY

Irwin I. Shapiro
Lincoln Laboratory,* Massachusetts Institute of Technology, Lexington, Massachusetts
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Recent advances in radar astronomy have made possible a fourth test of Einstein’s theory of general relativity. The test involves measuring the time delays between transmission of radar pulses towards either of the inner planets (Venus or Mercury) and detection of the echoes. Because, according to the general theory, the speed of a light wave depends on the strength of the gravitational potential along its path, these time delays should thereby be increased by almost $2 \times 10^{-4}$ sec when the radar pulses pass near the sun.1 Such a change, equivalent to 60 km in distance, could now be measured over the required path length to within about 5 to 10% with presently obtainable equipment.2

An analytical representation of this predicted increase in delay, useful for discussion, can be obtained by calculating the difference $\Delta t_r$ between the proper-time delay predicted in general relativity and the corresponding flat-space value. Using the usual form of the Schwarzschild solution to represent the gravitational field of the sun3 and neglecting the motion of the earth between pulse transmission and echo reception, we find

$$\Delta t = \frac{4\pi}{c} \left( \ln \frac{x + (x^2 + d^2)^{1/2}}{-x_e + (x^2 + d^2)^{1/2}} \right) - \frac{1}{2} \left[ \frac{x_p}{(x^2 + d^2)^{1/2}} + \frac{2x_e + x_p}{(x^2 + d^2)^{1/2}} \right] + O\left( \frac{x^2}{c^2} \right), \quad (1)$$

where, in this coordinate system, $d$ is the distance of closest approach of the radar wave to the center of the sun, $x_e$ is the distance along the line of flight from the earth-based antenna to the point of closest approach to the sun, and $x_p$ represents the distance along the path from this point to the planet. Both $x_e$ and $x_p$ are measured positively in a direction away from the earth. The gravitational radius $r_g$ for the sun is $GM_s/c^2=1.5$ km, where $G$ is the gravitational constant, $M_s$ the mass of the sun, and $c$ the speed of light. The right-hand side of Eq. (1) is due primarily to the variable speed of the light ray; the contribution from the change in path, being of second order in $(r_0/c)$, is negligible. (This type of result is a general one for refraction phenomena in which the change in index is small.)

At superior conjunction, when the target planet is on the opposite side of the sun from the earth, Eq. (1) reduces to

$$\Delta t = \frac{4\pi}{c} \left( \ln \frac{x + (x^2 + d^2)^{1/2}}{-x_e + (x^2 + d^2)^{1/2}} \right); \quad (2)$$

and, at inferior conjunction, when the planet is between the earth and sun, to

$$\Delta t = \frac{4\pi}{c} \left( \ln \frac{x_e}{x_p} - \frac{x_e - x_p}{2x_e} \right); \quad (3)$$

where $d \ll x_e, x_p$. At elongation, when the planet is furthest east...
or west from the sun, as viewed from the earth, we find

\[ \Delta t = \frac{4\gamma}{c} e^{\ln \left( \frac{2x}{d} \right) - 1}; \quad (4) \]

This last form is only valid for Mercury, since for Venus \( x_e = d \) at elongation. As an illustration, we note that for Mercury when \( d \) is twice the radius \( R_S \) of the sun, Eq. (2) yields \( \Delta t \approx 1.6 \times 10^{-4} \) sec, whereas for their respective conditions of applicability, Eqs. (3) and (4) both yield about \( 0.1 \times 10^{-4} \) sec. Thus, despite the logarithmic behavior of the dominant term in Eq. (1), the difference between the maximum and minimum effects, which is the significant measurable quantity, is almost equal to the maximum value of \( \Delta t \).

Are these effects on interplanetary time delays likely to be obscured by others? The most important candidates in this latter category are the imprecision in the knowledge of planetary orbits and radii, and the presence of the interplanetary medium. Analysis shows that the orbits of the earth and target planet, as well as the latter's radius, can be determined with more than the required precision from time-delay measurements distributed around the orbits of both planets. The sensitivity of the time-delay measurements to changes in \( \Delta t \) is different from the corresponding sensitivity to changes in the initial conditions of the orbits and in the planetary masses and radii. Hence the parameters characterizing \( \Delta t \) can be estimated from the data simultaneously with the other relevant ones, without incurring any severe accuracy penalty from nonseparability.

The topographical variations on the target planets are probably small enough so that even the most accurate measurements will not be significantly degraded.

The effect \( \Delta t_m \) of the interplanetary medium on the time delay can be represented by

\[ \Delta t_m = 8.2 \times 10^7 \int_{\pi} e_0 \left( N(l) \right) dl \text{ sec,} \quad (5) \]

where \( N \) is expressed in electron/cm\(^3\), \( f \) in cps, \( c \) in cm/sec, and \( l \) in cm. Using recently compiled results on the solar corona, we find that during a "quiet-sun" period

\[ N(r) = 5 \times 10^9 \left( \frac{R_S}{r} \right)^2 \text{ electron/cm}^3, \quad r^2 = t^2 + d^2, \quad (6) \]

represents the data reasonably well from about \( r = 4R_S \) to \( r = 20R_S \). Inside this range, the actual \( N \) increases more rapidly with decreasing \( r \), whereas outside it decreases more rapidly with increasing \( r \). For the period of maximum solar activity, \( N \) seems to be about a factor of 5 higher in the radial range represented by (6). Substituting Eq. (6) into (5) yields

\[ \Delta t_m = 6.5 \times 10^{24} \int \left( \tan^{-1} \left( \frac{x}{d} \right) + \tan^{-1} \left( \frac{x}{e} \right) \right) \text{ sec,} \quad (7) \]

where \( d \), \( x_e \), and \( x_p \) are expressed in cm. For the Arecibo Ionospheric Observatory's frequency of 430 Mc/sec, the lowest at which interplanetary time-delay measurements are currently being made, Eq. (7) yields \( \Delta t_m = 3.7 \times 10^{-4} \) sec for observations of Mercury near superior conjunction with \( d = 4R_S \). This latter value corresponds to an angular distance from the sun of 1°, the smallest at which Arecibo measurements can be made (see below). In this case, \( \Delta t_r \) would equal about \( 1.4 \times 10^{-4} \) sec and would most likely be masked by the uncertainty in \( \Delta t_m \). Although \( \Delta t_m \) varies inversely with \( d \), whereas the corresponding dependence in \( \Delta t_r \) is logarithmic, the difference \( \Delta t_r - \Delta t_m \) is nowhere large enough and positive for a really reliable result to be obtained solely from Arecibo data. Since \( \Delta t_m \) varies as the inverse square of the radar frequency, this plasma effect will be reduced by a factor of almost 400 (and will therefore be unimportant) for measurements made at the 8350-Mc/sec frequency of the newly constructed, but not yet fully instrumented, Haystack radar of MIT's Lincoln Laboratory. In any event, simultaneous equivalently accurate time-delay measurements at two frequencies will allow the plasma effect to be deduced and subtracted, since \( \Delta t_m \) is frequency dependent and \( \Delta t \) is not.

Other possibly relevant effects on the delays are easily disposed of. A previous study has shown that the earth’s and planet’s atmospheres and ionospheres will not significantly affect time delays, even for \( f = 430 \text{ Mc/sec} \). The effect of the earth’s gravity and motion on the laboratory clock is unimportant for this experiment, since the clock rate remains constant over a year to within about one part in \( 10^{10} \). The gravitational effects of the earth, moon, and target planet on the time delays are far smaller than the sun’s, but in any case the former can be neglected since their contributions will be almost identical in each measurement.
and consequently indistinguishable from a small
decrease in the planet's radius. Any lack of
precision in the determination of \( c \) in terms
of terrestrial units (such as km/sec) is clear-
ly irrelevant to our experiment since time de-
lays only are of concern.

In making the time-delay measurements, a
radar cannot be directed towards the solar
limb because of the radio interference that
would result. For the Haystack facility, how-
ever, the antenna beam width is sufficiently
narrow and the near side lobes of sufficiently
low gain that the beam can be directed well
within a degree of the limb without the solar
radio emanations introducing a significant in-
crease in the overall system noise temperature.
For Arecibo the closest possible approach is
about 1°. Were both inner planets to have the
same ratio of radar to geometric cross section,
then at superior conjunction Venus would al-
ways be more easily detected, the received
power being about a factor of two greater. How-
ever, at the X-band frequency of Haystack,
Venus may very well have a relatively low ra-
dar cross section\(^\text{10}\) because of absorption in
its atmosphere.

Although Mercury passes through superior
conjunction about once in four months, some
passages are considerably more useful than
others from the points of view of ease of detect-
ability and of close angular approach to the sun.
The most favorable in the next two years occur
on 11 June 1965 and 27 May 1966; in the former
the minimum angular distance is about 1° and
in the latter about 0.5°. The superior conjunc-
tions of Venus are less variable and usually
lead to an angular distance of closest approach
of about 1°, as in the next two which will be on
12 April 1965 and 9 November 1966. The sec-
ond occurs in southern declinations and so will
be invisible to the Arecibo radar whose anten-
a is not steerable.

\^{*}\text{Operated with support from the U. S. Air Force.}
\^{1}In an interplanetary radar experiment, the Dop-
pler shift of the radar wave is also measured; but
although the effect on time delay of the change in
\( c \) is cumulative, the corresponding general rela-
\text{tivistic effect on Doppler cancels out over the round
trip.}
\^{2}J. V. Evans and G. H. Pettengill, private com-
munication.
\^{3}See, for example, P. G. Bergmann, Introduction
to the Theory of Relativity (Prentice-Hall, Inc.,
\^{4}In an earth-moon experiment, however, the rela-
tivistic contribution to the difference in measured
time delay between new and full moon is quite un-
detectable, being only about \( 5 \times 10^{-11} \) sec.
\^{5}With measurements extended over a several-
year period the precession of Mercury's perihelion
could also be estimated accurately and indepen-
dently of the optical data; in addition, reasonably
strict limits could be placed on the possible time
dependence of \( G \).
\^{7}The only other facility that could now participate
significantly in this measurements program is Jet
Propulsion Laboratory's Goldstone radar, op-
\text{erated at a frequency of 2388 Mc/sec.}
\^{8}With one of these frequencies at about 400 Mc/sec
or lower, the integrated electron density along
the path of the radar wave could be determined
accurately, thus providing useful information on
the solar corona. Faraday rotation effects may
also shed light on certain characteristics of the
solar magnetic field.
\^{9}J. I. Shapiro, to be published.
\^{10}D. Karp, W. E. Morrow, Jr., and W. B. Smith,
to be published.

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**ION TRAPPING IN ROTATING HELIUM II**

R. L. Douglass

Physics Department, American University of Beirut, Beirut, Lebanon

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Trapping of negative ions in the cores of quan-
tized vortex lines has been suggested by Car-
eri, McCormick, and Scaramuzzi\(^\text{1}\) as an expla-
nation for their discovery of rotation-induced
attenuation of space-charge limited ion currents
in He II. This Letter reports the direct detect-
tion of trapped negative ions in rotating He II,
and some measurements of their mean trapped
time and of their mobility parallel to the rota-
tion axis at temperatures between 1.20 and
1.72°K. No evidence of positive-ion trapping
has been found, in agreement with the ion-cur-
rent measurements\(^\text{1,2}\) and in contrast to the
experiment of Rayfield and Reif\(^\text{3}\) at much lower
temperatures.

The He II is contained in an electrode assem-