Nonrelativistic contribution to Mercury's perihelion precession

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We present here a calculation of the precession of the perihelion of Mercury due to the perturbations from the outer planets. The time-average effect of each planet is calculated by replacing that planet with a ring of linear mass density equal to the mass of the planet divided by the circumference of its orbit. The calculation is easier than examples found in many undergraduate theoretical mechanics books and yields results which are in excellent agreement with more advanced treatments. The perihelion precession is seen to result from the fact that the outer planets slightly change the radial period of oscillation from the simple harmonic period usually calculated for small displacements from equilibrium. This new radial period therefore no longer matches the orbital period and the orbit consequently does not exactly retrace itself. The general question of whether a given perturbation will cause the perihelion to advance or regress is shown to have the following answer: if a perturbing force is central and repulsive and also becomes stronger as the distance from the force center increases, the perihelion will advance. If the central perturbing force is attractive and also becomes stronger as the distance from the force center increases, the perihelion will regress.

I. INTRODUCTION

The precession of the perihelion of the planet Mercury is a topic which arises frequently in discussions concerning general relativity[1] since this precession is one of the three direct observational foundations on which the theory rests. It is shown in most mechanics books[2-8] that the simple theory of planetary motion in a central inverse square gravitational field predicts that the planetary orbits are ellipses with the sun at one focus. In this simple theory, the major axis of the ellipse remains fixed in space so that the point of closest approach between the planet and the sun (perihelion) does not move. Most authors of texts on mechanics,[2-8] astronomy,[9,10] and relativity[1] mention that the major axis of the orbit of Mercury is observed to shift in space as a result of both perturbations from the outer planets (by 532 arc sec per century) and from relativistic effects (by 42.8 arc sec per century). Since the relativistic effects are usually the major topic of interest, the planetary perturbations are dealt with by simply quoting the result of an unreferenced calculation and then considering the relativistic calculation.

It is our purpose here to show that the component of the precession of Mercury's perihelion due to outer planets can be calculated easily at the undergraduate level. The computation can be done as an example worked in class or—with some hints—assigned as a homework problem. More specific suggestions are included in Sec. V. Used in this way, the calculation presented here provides an excellent exercise in classical mechanics while simultaneously preparing the student for material to be encountered in more advanced courses.

Our approach follows closely that taken by Fowles[6] and Symon,[5] both of whom consider the effect of a ring of matter on a planetary orbit. However, both authors consider the matter ring to be interior to the planet, making their results inapplicable to the planet Mercury. We note in passing that the addition of an interior ring is equivalent to a quadrupole mass moment for the sun and causes a precession.

Intuitively, the justification for replacing a planet by a ring of mass is that the force experienced by Mercury due to the Sun is much greater than that due to the exterior planets. Consequently, Mercury's deviation from its unperturbed orbit is very small during times on the order of the period of an exterior planet's orbit. Hence, a mass ring is an approximate way of representing the time averaged effect of a moving planet. While this may seem to be a reasonably accurate approximation, one should bear in mind that it neglects possible orbital resonance effects which can occur if the period of one planet is a small rotational fraction of the period of the other.

In the calculation which follows, we first calculate the force field inside of a uniform ring of matter and in the plane of the ring. This is equivalent to replacing each of the outer planets by a ring whose linear mass density, λ_i, is

$$\lambda_i = M_i/2\pi R_i, \tag{1}$$

where M_i is the mass of the outer planet and R_i in the radius of its orbit. We then consider the effect of this small perturbing force on the orbit of Mercury by examining the period of oscillation of Mercury about a circular orbit. We find that if Mercury is slightly disturbed from a circular orbit, the period of oscillation about the average radius will not be equal to the orbital period, resulting in a slow precession of the perihelion.

II. GRAVITATIONAL FIELD OF A UNIFORM RING

To begin, we approximate the time-averaged gravitational field of each planet exterior to Mercury as a uniform circular ring centered on the sun and in the plane defined by the orbit of Mercury. Symbols are defined in Fig. 1. A point mass m is on line ABC at a distance a from the center

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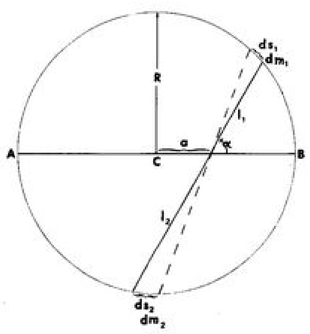

Fig. 1. The symbols used in calculating the gravitational forces on mass m, located at distance a from the center of a uniform ring of linear mass density λ, are illustrated here.

C of the ring of radius R. To compute the force acting on m, consider breaking the ring up into differential mass elements, dm_1 and dm_2, located at angle α from line AB as shown in Fig. 1. If ds_1 and ds_2 are the arcs subtended by differential angular element $d\alpha$, and if a is small compared to R,[11]

$$dm_i = \lambda ds_i \simeq \lambda l_i d\alpha, \tag{2}$$

where i may be 1 or 2 and l_i is the distance from m to the ring. Newton's law of gravitation gives us the net force on m as

$$d\mathbf{F} = Gm\left(\frac{dm_1}{l_1^2} - \frac{dm_2}{l_2^2}\right)\hat{l}, \tag{3}$$

where $\hat{l}$ is a unit vector from m to dm_1. Substituting Eq. (2) into (3) we find

$$d\mathbf{F} = mG\lambda\left(\frac{l_2 - l_1}{l_1 l_2}\right)\hat{l}\, d\alpha.$$

From symmetry we see that only components of $d\mathbf{F}$ along line AB affect m since components perpendicular to AB will cancel. Thus, we have a radial force, $d\mathbf{F}_r$ given by

$$d\mathbf{F}_r = d\mathbf{F}\cos\alpha.$$

If r is a unit vector in the radial direction, we may integrate to obtain

$$\mathbf{F} = \hat{r}\int_{-\pi/2}^{\pi/2} mG\lambda\left(\frac{l_2 - l_1}{l_1 l_2}\right)\cos\alpha d\alpha, \tag{4}$$

the integration limits having been chosen so that the entire ring is considered. The α dependence of l_1 and l_2 can be made explicit by applying the law of cosines which gives us

$$R^2 = a^2 + l_1^2 - 2al_1\cos(\pi - \alpha).$$

Solution of this quadratic equation for l_1 yields

$$l_1 = -a\cos\alpha \pm [a^2\cos^2\alpha - (a^2 - R^2)]^{1/2}.$$

The correct root may be selected by noting the physical requirement for $\alpha = 0$, l_1 must be equal to $R - a$; thus, we select the positive value of the radical. Repeating this procedure for l_2, we again must chose the positive radical resulting in

$$l_2 = a\cos\alpha + [a^2\cos^2\alpha - (a^2 - R^2)]^{1/2}.$$

Substituting the expressions for l_1 and l_2 into Eq. (4), the integral trivially reduces to

$$\mathbf{F} = \frac{2G\lambda am}{R^2 - a^2}\hat{r}\int_{-\pi/2}^{\pi/2}\cos^2\alpha d\alpha.$$

Doing the integration yields

$$\mathbf{F} = [mG\pi\lambda a/(R^2 - a^2)]\hat{r}. \tag{5}$$

Thus, the ring provides a force field equivalent to a central repulsive force whose functional form is

$$F(r)\alpha[r/(R^2 - r^2)].$$

This small equivalent repulsive force will be added to the much larger attractive force due to the sun.

III. ORBITAL CLOSURE

It is shown in most intermediate mechanics books[4-7] that any attractive central force can produce a circular orbit but that such orbits need not be either stable or closed. The circular orbit is stable if a small radial displacement results in oscillation about the original radial distance. The orbit is closed if the period of radial oscillation is some rational multiple of the orbital period so that the orbiting body eventually retraces its own orbital path. Following Fowles,[6] and Becker,[7] we note that if $\Phi(r)$ is the total central force, the equation of motion in the radial direction is

$$\Phi(r) = m(\ddot{r} - \dot{\theta}^2 r), \tag{6}$$

where a dot implies time differentiation. The last term in Eq. (6) has the physical interpretation of being a centrifugal force. Since angular momentum J is a constant of the motion, we have

$$J = mr^2\dot{\theta}. \tag{7}$$

Solving for $\dot{\theta}$ and substituting into Eq. (6) yields

$$\Phi(r) = m(\ddot{r} - J^2/m^2r^3), \tag{8}$$

where the physical interpretation of $\Phi(r)$ is that it is the total applied force.

For the special case of a circular orbit of radius a, $\ddot{r}$ is zero and Eq. (8) reduces to

$$\Phi(a) = -J^2/(ma^3). \tag{9}$$

If the planet is now disturbed slightly in the plane of its orbit and normal to its initial path, it will oscillate about a. We define $X \equiv r - a$ and express the radial equation of motion in terms of X. Thus,

$$\begin{aligned}\Phi(X + a) &= m\ddot{X} - J^2m^{-1}(X + a)^{-3}\\ &= m\ddot{X} - J^2m^{-1}a^{-3}(1 + X/a)^{-3}.\end{aligned}$$

Since X/a is much less than unity, we may use the binomial expansion on the term in parenthesis, retaining only first-order terms. If we expand the left-hand side in a Taylor series about the point $r = a$ and again retain only first-order terms, we may rewrite our last equation as

$$\Phi(a) + \Phi'(a)X = m\ddot{X} - (J^2/ma^3)(1 - 3X/a).$$

Here a prime denotes differentiation with respect to X. If we substitute Eq. (9) into the above equation, we find

$$\ddot{X} + (1/m)[-(3/a)\Phi(a) - \Phi'(a)]X = 0. \tag{10}$$

Note that this equation describes a simple harmonic oscillator if the term in brackets is positive. (If the term is negative, we have an exponential solution and the orbit is unstable.) Thus, for stable orbits, the period of oscillation

Table I. Planetary data and calculations.

Planet number	Planet name	Mass (10^{24} kg) (Ref. 13)	Orbit radius (10^{11} m) (Ref. 12)	λ (10^{12} kg m^{-1})	$\lambda a/(R^2-a^2)$ (kg m^{-2})	$\lambda a/(R^2-a^2)$ $(R^2+a^2)/(R^2-a^2)$ (kg m^{-2})
1	Mercury	0.3332	0.5791	...	...	...
2	Venus	4.870	1.082	7.163	49.65	89.51
3	Earth	5.976	1.496	6.358	19.35	26.17
4	Mars	0.6421	2.279	0.4484	0.5345	0.6083
5	Jupiter	1899	7.783	388.3	37.33	37.75
6	Saturn	568.6	14.27	63.42	1.807	1.813

about $r = a$ will be

$$\tau = 2\pi \left(\frac{m}{-(3/a)\Phi(a) - \Phi'(a)}\right)^{1/2}. \qquad (11)$$

By definition, an apsis is a point in an orbit at which the radius vector assumes an extreme value and the apsidal angle ψ is the angle swept out by the radius vector between two consecutive apsides. The time required for Mercury to sweep out this angle is clearly $\tau/2$. Since r has the approximately constant value a, we can solve Eq. (7) for $\dot{\theta}$ and write

$$\psi = \frac{1}{2}\tau\dot{\theta} = \frac{1}{2}\left[2\pi\left(\frac{m}{-(3/a)\,\Phi(a) - \Phi'(a)}\right)^{1/2}\right]\left(\frac{J}{ma^2}\right).$$

Rearranging Eq. (9), we note that the last term in the above equation is

$$J/ma^2 = [-\Phi(a)/ma]^{1/2},$$

so that

$$\psi = \pi\{3 + a[\Phi'(a)/\Phi(a)]\}^{-1/2}. \qquad (12)$$

If $\Phi(a)$ describes the gravitational field due only to a point mass sun,

$$F_0(r) = -GM_0mr^{-2}, \qquad (13)$$

substitution of this into Eq. (12) yields the expected result

$$\psi = \pi. \qquad (14)$$

In short, the orbit in a pure inverse square field may be viewed as repeating itself exactly because the period of oscillation about a circular orbit happens to equal the orbital period.

IV. COMPUTATIONS AND RESULTS

Using Eq. (5) we may now evaluate the forces acting on Mercury due to the outer planets. Since the time averaged effect is being approximated by rings and we have found that the force is radial, Eq. (5) gives the radial outward force due to the rings as

$$F(a) = G\pi m \sum_{i=2}^{9} \lambda_i \frac{a}{R_i^2 - a^2}. \qquad (15)$$

It will be recalled that m and a are the mass and orbital radius of Mercury. Numerical values are shown in Table I. Since the term following the summation sign in Eq. (15) is a measure of the relative influence of each outer planet on Mercury, inspection of the next to last column of Table I shows that the planets beyond Saturn may be neglected. These outermost planets are less massive and far more distant than is Saturn. Substituting in numerical values, we find

$$F(a) = 7.587 \times 10^{15}\ \text{N}.$$

From Eq. (13), the force F_0 due to the sun on Mercury is

$$F_0 = -1.318 \times 10^{22}\ \text{N}. \qquad (16)$$

The total applied force is

$$\Phi(a) = F_0 + F(a). \qquad (17)$$

Evaluating the derivative term required in Eq. (12), we find

$$a\Phi'(a) = a\,[F_0' + F'(a)].$$

Substituting Eqs. (13) and (15) into the expression and neglecting planets more distant than Saturn, we arrive at

$$a\Phi'(a) = -2F_0 + Gm\pi aS, \qquad (18)$$

where we have defined

$$S \equiv \sum_{i=2}^{6} \lambda_i \frac{R_i^2 + a^2}{(R_i^2 - a^2)^2}, \qquad (19)$$

where this function is the derivative of the term in Eq. (15). Substituting the result into Eq. (12) yields

$$\psi = \pi\left(3 + \frac{[-2F_0 + Gm\pi aS]}{F_0 + F(a)}\right)^{-1/2},$$

which may be rewritten

$$\psi = \pi\left(\frac{1 + [3F(a) + G\pi maS]/F_0}{1 + [F(a)/F_0]}\right)^{-1/2}. \qquad (20)$$

We may use the binomial expansion for both the numerator and demoninator of Eq. (20). Since $F(a) \ll F_0$, and $F(a)$ is of the same order of magnitude as $G\pi maS$, we may neglect terms of higher order than first power of the ratio $F(a)/F_0$. Eq. (20) may then be written

$$\psi = \pi\left(1 - \frac{3F(a) + G\pi maS}{2F_0}\right)\left(1 + \frac{F(a)}{F_0}\right).$$

If we carry out the indicated multiplication and neglect second-order terms, we find

$$\psi = \pi\left(1 - \frac{F(a)}{F_0} - \frac{G\pi maS}{2F_0}\right).$$

Substituting numerical values gives us

$$\psi = \pi(1 + 9.884 \times 10^{-7}).$$

Since we have defined ψ as the angle between perihelion and aphelion, it is reasonable to define the rate of precession of the perihelion $\dot{\omega}$ as

$$\dot{\omega} \equiv \frac{2\psi - 2\pi}{P} = \frac{\pi(1.977 \times 10^{-6})}{87.969 \text{ days}},$$

where P is the sidereal period of Mercury. On converting this to more conventional units, we find

$$\dot{\omega} = 531.9 \text{ arc sec/century}. \qquad (21)$$

The positive value indicates a perihelion advance. This result compares quite well with the results of more advanced treatments[1] which show that the total dynamical precession of 575 sec of arc per century consists of a planetary perturbation component of 532 sec of arc per century with a relativistic contribution of 43 sec of arc per century.

V. DISCUSSION

If one inquires into the physical explanation of the precession phenomenon, several points of interest arise. Since the outer planets have been replaced by equivalent solid rings, the direction of their motion is unimportant. The physical reason that the major axis of the orbit precesses is that the addition of the small equivalent central repulsive force described by Eq. (5) alters the period of oscillation about a circular orbit so that the orbital and radial periods are not quite equal, resulting in a slow precession. Note that whether the perihelion point advances (moves in the same direction as the planet) or regresses depends on whether the ratio $a\Phi'/\Phi$ in Eq. (12) is smaller or larger than -2, respectively. An interesting discussion of the general question of the conditions under which no precession will occur has recently been given by Brown.[14]

Replacing the total applied force with the gravitational and perturbing terms, the condition for the perihelion advance becomes

$$a[F_0' + F'(a)]/[F_0 + F(a)] < -2.$$

Resulting from Eq. (13),

$$F_0' = 2G\pi m a^{-3} = -(2/a)F_0.$$

Making these substitutions,

$$[-2F_0 + aF'(a)]/[F_0 + F(a)] < -2$$

or that

$$aF'(a) + 2F(a) > 0.$$

(Note that the change in sign for the inquality results from the fact that F_0 is negative).

It is now evident that if a perturbing force is central and repulsive and also becomes stronger as the distance from the force center increases, the perihelion will advance. If the central perturbing force is attractive and also becomes stronger as the distance from the force center increases, the perihelion will regress.

We will make the following suggestions for using this problem in various physics classes: Instructors in relativity courses who wish to review the observational support for general relativity can either present this example, assign it as a problem, or reference it. In classical mechanics courses, an instructor may wish to work or assign for reading the example of precession due to a ring of matter interior to the orbit of Mercury, while assigning this example as a homework problem or doing it as an example not presented in the reading. We note that including examples of rings both interior and exterior to Mercury provides students with a thorough preparation for general relativity. A ring interior to a planet gives rise to an r^{-4} force which represents either the quadrupole mass moment of the sun[5] or the general relativistic correction term.[4] As discussed above, the exterior ring problem permits computation of the classical precession terms.

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[9] L. Motz and A. Duveen, *Essentials of Astronomy*, 2nd ed. (Columbia University, New York, 1977), p. 155.

[10] E. v. P. Smith and K. C. Jacobs, *Introductory Astronomy and Astrophysics* (Saunders, Philadelphia, 1973), p. 154.

[11] The astute reader will note that Eq. (2) will not be valid for values of $a \sim R$, since the angle $d\alpha$ would subtend a mass element larger than ds_i by a factor $(\cos\alpha)^{-1}$. Since the orbits of Venus and Mercury differ by approximately a factor of 2, we repeated the calculation including the $(\cos\alpha)^{-1}$ term. An exact solution could not be found, but a series expansion to terms of order (a/R) showed that the errors introduced by ignoring the $(\cos\alpha)^{-1}$ term were of the order of 2.3%.

[12] Reference 9, p. 227.

[13] *The Cambridge Encyclopedia of Astronomy*, edited by S. Mitton (Crown, New York, 1977), pp. 173 and 206.

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