Relativity of simultaneity in SR

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1 Einstein's original paper

Shown below is §2 of Einstein's paper: **On the Electrodynamics of Moving Bodies**

§ 2. On the Relativity of Lengths and Times

The following reflexions are based on the principle of relativity and on the principle of the constancy of the velocity of light. These two principles we define as follows:--

1. The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of co-ordinates in uniform translatory motion.
2. Any ray of light moves in the ``stationary" system of co-ordinates with the determined velocity c, whether the ray be emitted by a stationary or by a moving body. Hence

$$\text{velocity} = \frac{\text{light path}}{\text{time interval}}$$

where time interval is to be taken in the sense of the definition in § 1.

Let there be given a stationary rigid rod; and let its length be l as measured by a measuring-rod which is also stationary. We now imagine the axis of the rod lying along the axis of x of the stationary system of co-ordinates, and that a uniform motion of parallel translation with velocity v along the axis of x in the direction of increasing x is then imparted to the rod.

We imagine further that at the two ends A and B of the rod, clocks are placed which synchronize with the clocks of the stationary system, that is to say that their indications correspond at any instant to the ``time of the stationary system" at the places where they happen to be. These clocks are therefore ``synchronous in the stationary system."

We imagine further that with each clock there is a moving observer, and that these observers apply to both clocks the criterion established in § 1 for the synchronization of two clocks. Let a ray of light depart from A at the time t_A, let it be reflected at B at the time t_B, and reach A again at the time t'_A. Taking into consideration the principle of the constancy of the velocity of light we find that

$$t_B - t_A = \frac{r_{AB}}{c - v} \text{ and } t'_A - t_B = \frac{r_{AB}}{c + v}$$

where r_{AB} denotes the length of the moving rod--measured in the stationary system. Observers moving with the moving rod would thus find that the two clocks were not synchronous, while observers in the stationary system would declare the clocks to be synchronous.

2 Elaboration of Einstein's thought experiment

At the risk of offending the intelligent reader, I will repeat Einstein's proof that the relativity of simultaneity follows from the postulates of SR.

Two clocks, A and B, are placed at the ends of a rod. This rod is moving with the speed v in the 'stationary system'. The length of the rod is r_{AB} as measured in the stationary system. Einstein specifies that the indication of these clocks "correspond at any instant to the 'time of the stationary system' at the places where they happen to be". The 'time of the stationary system' (coordinate clocks of the stationary system) are synchronous according to the definition in §1, and this is Einstein's way of making the moving clocks A and B synchronous in the stationary system.

Note that this means that the clocks on the rod can't be equal to the clocks in the stationary system; their intrinsic rate has to be slightly higher. This has no significance for the thought experiment, though, the indications of the clocks in the stationary system are never used.

So we have:
A rod with two clocks A and B at its ends is moving with speed v in the stationary system.
The length of the rod is r_{AB} as measured in the stationary system.
The clocks A and B are synchronous in the stationary system.

The question is now, are they synchronous in the rest frame of the rod (moving system)?

To determine that, a light pulse is emitted from clock A, reflected off clock B, and received by A.

Let clock A show t_A at when the light is emitted.

The light will be reflected off clock B when clock B shows $t_B = t_A + t_{AB}$, where t_{AB} is the transit time for the light to go from clock A to clock B, as measured in the stationary system. Since clock B will recede a distance vt_{AB} while the light is in transit, the path length of the light will be $r_{AB} + vt_{AB}$. Since the speed of light is c in the stationary system, we have: $ct_{AB} = r_{AB} + vt_{AB}$. So the transit time $A \rightarrow B$ is:

$$t_{AB} = \frac{r_{AB}}{c - v} \tag{1}$$

The light will be received by clock A when clock A shows $t'_A = t_B + t_{BA}$ where t_{BA} is the transit time for the light to go from clock B to clock A, as measured in the stationary system. Since clock B will approach a distance vt_{AB} while the light is in transit, the path length of the light will be $r_{AB} - vt_{AB}$. Since the speed of light is c in the stationary system, we have: $ct_{BA} = r_{AB} - vt_{BA}$. So the transit time $B \to A$ is:

$$t_{BA} = \frac{r_{AB}}{c + v} \tag{2}$$

We have $t_{AB} = t_B - t_A$ and $t_{BA} = t'_A - t_B$ so:

$$t_B - t_A = \frac{r_{AB}}{c - v} \tag{3}$$

$$t'_A - t_B = \frac{r_{AB}}{c + v} \tag{4}$$

In the rest frame of the rod, the clocks A and B are stationary, and since the speed of light is c in the rest frame of the rod, the criterion for stationary clocks to be synchronous is: $(t_B - t_A) = (t'_A - t_B)$. But $(t_B - t_A) > (t'_A - t_B)$ so A and B are not synchronous in the rest frame of the rod.

Clock A and clock B do not simultaneously show the same as observed in their rest frame.

3 Conclusion

Einstein's thought experiment shows that an inevitable consequence of the postulates of SR is that simultaneity is relative.