Radial speed LLR observatory - Moon as a function of the relative position of the Moon

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1 Calculation of the radial speed of the Moon as observed from an observatory on the Earth

We will in the following make the rather crude assumption that the Moon's orbit is a circle in the equatorial plane.

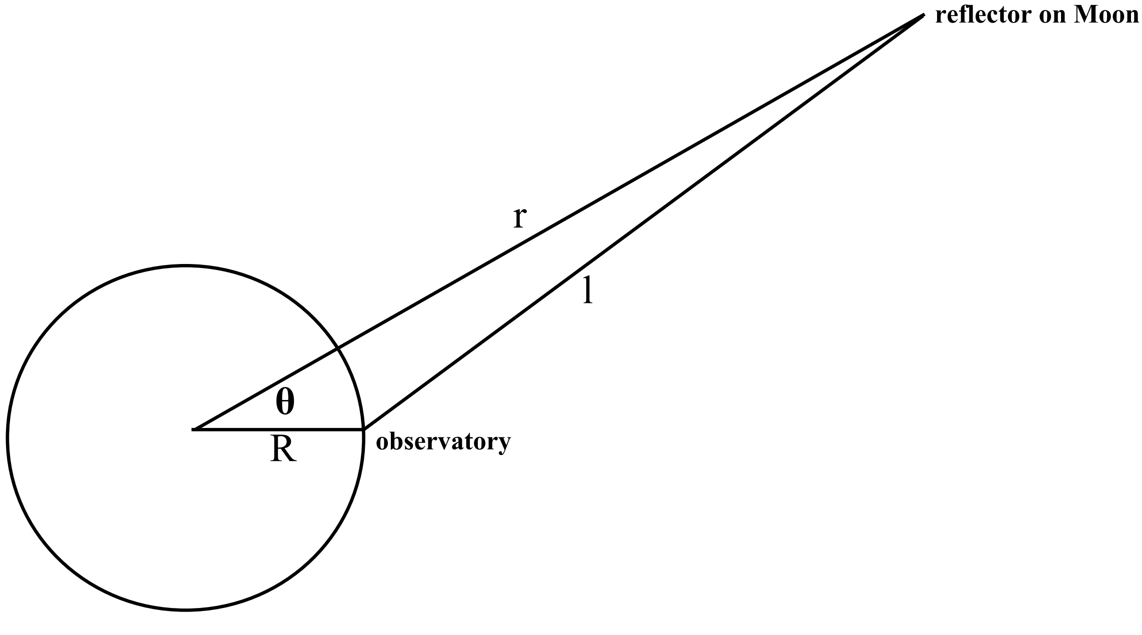

Figure 1: *System seen from north celestial pole*

Elementary geometry yields:

$$l = \sqrt{R^2 + r^2 - 2rR\cos\theta} \tag{1}$$

where:

l is the distance from the observatory to the reflector on the Moon
R is the radius vector from the centre of the Earth to the observatory

$\boldsymbol{r}$ is the radius vector from the centre of the Earth to the reflector on the Moon
$\boldsymbol{\theta}$ is the angle between $\vec{R}$ and $\vec{r}$

Differentiation yields:

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \frac{rR\sin\theta}{\sqrt{R^2 + r^2 - 2rR\cos\theta}}\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{2}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega - \Omega \tag{3}$$

where ω is the angular velocity of the Earth and Ω is the angular velocity of the Moon.

Since ω and Ω are constants, the radial speed of the Moon relative to the observatory as a function of θ is:

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \frac{rR\,(\omega - \Omega)\sin\theta}{\sqrt{R^2 + r^2 - 2rR\cos\theta}} \tag{4}$$

This equation is for an observatory at equator. For an observatory at another latitude, the equation becomes:

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \frac{rR\,(\omega - \Omega)\sin\theta\cos\beta}{\sqrt{R^2 + r^2 - 2rR\cos\theta}} \tag{5}$$

where β is the latitude of the observatory.

θ is generally the angle between the meridian and the position of the Moon in the equatorial plane. For an observatory at equator, θ will be the angle from zenith. θ is positive when the Moon is east of the observatory, and negative when it is west of it.

Since $r \gg R$, a first order approximation can be written:

$$\frac{\mathrm{d}l}{\mathrm{d}t} \approx R\,(\omega - \Omega)\left(1 + \frac{R}{r}\cos\theta\right)\sin\theta\cos\beta \tag{6}$$

Inserting the values:

$R \approx 6.4 \cdot 10^6\ m$
$r \approx 3.844 \cdot 10^8\ rad/s$
$\omega \approx 2.6617 \cdot 10^{-6}\ rad/s$
$\Omega \approx 7.292 \cdot 10^{-5}\ rad/s$

the equation becomes:

$$\frac{\mathrm{d}l}{\mathrm{d}t} \approx -450\,(1 + 0.017\cos\theta)\sin\theta\cos\beta\ \ m/s \tag{7}$$

The error in the approximation is less than 1‰ for all θ and β.

Note that the Moon is approaching when it is east of the observatory, and is receeding when it is west of it.

Fig. 2 shows the radial speed of the Moon as function of θ for an observatory at equator (green curve) and an observatory at latitude N40^0 (red curve).

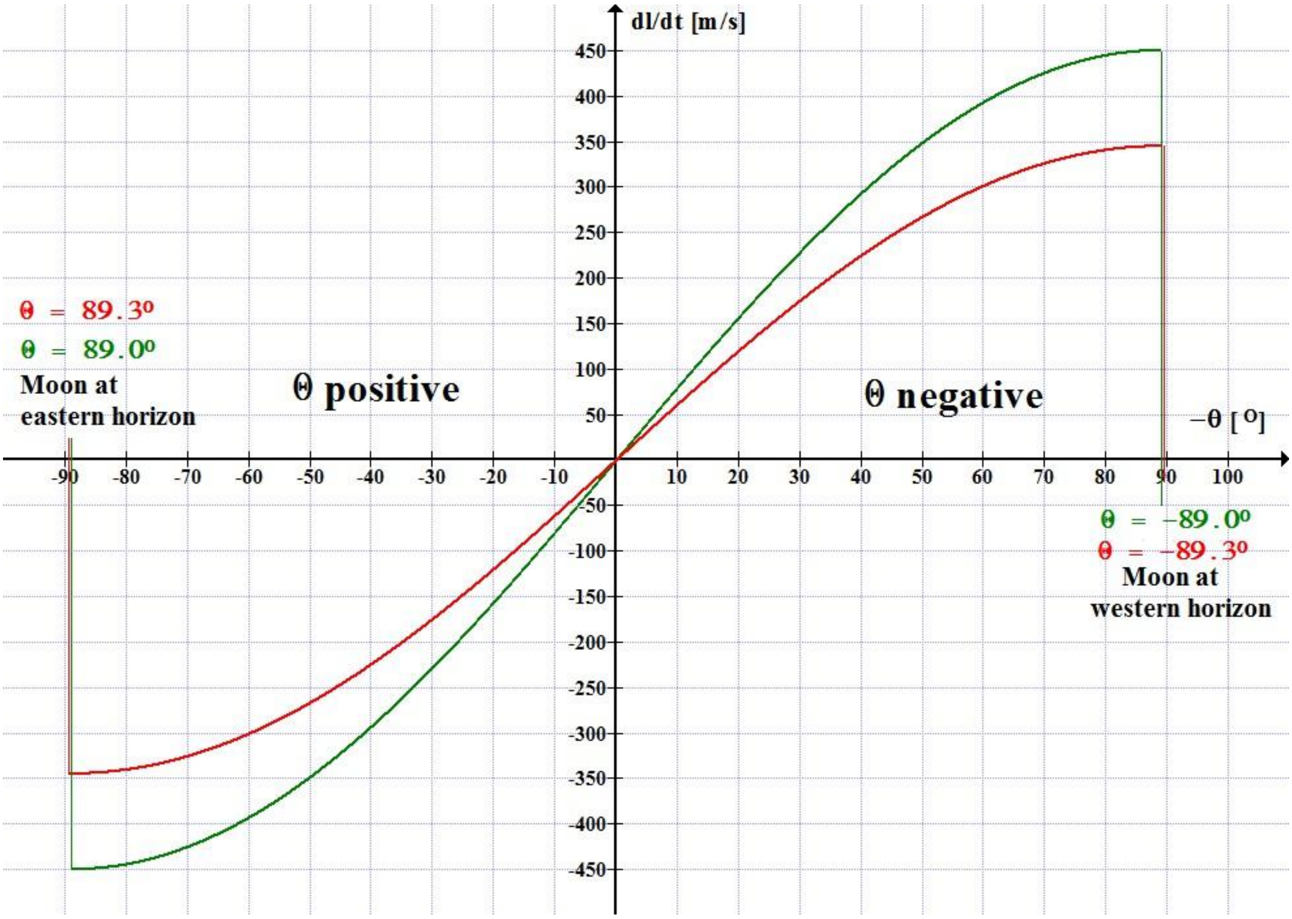

Figure 2: *Radial speed of the Moon as function of* θ

When the Moon moves from the eastern horizon to the western horizon,
the angle θ decreases from 90^0 to -90^0.

For an observatory at the equator:

When $\theta = 89.0^0$	the Moon is at the eastern horizon,	radial speed -447.4 m/s
when $\theta = 0^0$	the Moon is at zenith,	radial speed 0 m/s
when $\theta = -89.0^0$	the Moon is at the western horizon,	radial speed 447.4 m/s

For an observatory at latitude N40^0:

When $\theta = 89.3^0$	the Moon is at the eastern horizon,	radial speed -342.7 m/s
when $\theta = 0^0$	the Moon is at the meridian,	radial speed 0 m/s
when $\theta = -89.3^0$	the Moon is at the western horizon,	radial speed 342.7 m/s

Radial speed of the Moon caused by the eccentricity of the orbit

Since the eccentricity of the lunar orbit is quite small, an approximation of the geocentric distance Earth-Moon can be written:

$$l_g = \frac{l_a + l_p}{2} + \frac{l_a - l_p}{2} \cos \omega t \tag{8}$$

where:

$\boldsymbol{l_g}$ is the geocentric distance Earth-Moon
$\boldsymbol{l_a}$ is the geocentric distance Earth-Moon at apogee
$\boldsymbol{l_p}$ is the geocentric distance Earth-Moon at perigee
$\boldsymbol{\omega}$ is mean angular velocity of the Moon
$\boldsymbol{t}$ is time since apogee

Differentiating yields:

$$\frac{\mathrm{d}l_g}{\mathrm{d}t} = -\frac{(l_a - l_p)\,\omega}{2} \sin \omega t \tag{9}$$

Because of the influence of the Sun, the distances at apogee and perigee will vary considerably during the year, see: **Wikipedia: Lunar distance**

We will use these values:

$l_a = 4.05696 \cdot 10^8 \ m$
$l_p = 3.63104 \cdot 10^8 \ m$
$\omega = 2.6617 \cdot 10^{-6} \ rad/s$

With these values the equation becomes:

$$\frac{\mathrm{d}l_g}{\mathrm{d}t} = -56.68 \cdot \sin\left(2.6617 \cdot 10^{-6} t\right) \ m/s \tag{10}$$

Note that $t = 0$ at apogee.

The error in this approximation will be less than 1% for all t.

To find the radial speed of the Moon as observed from an observatory, the results from equation (7) and equation (10) must be added.

Fig. 3 shows the change of the geocentric distance Earth-Moon with respect to time.

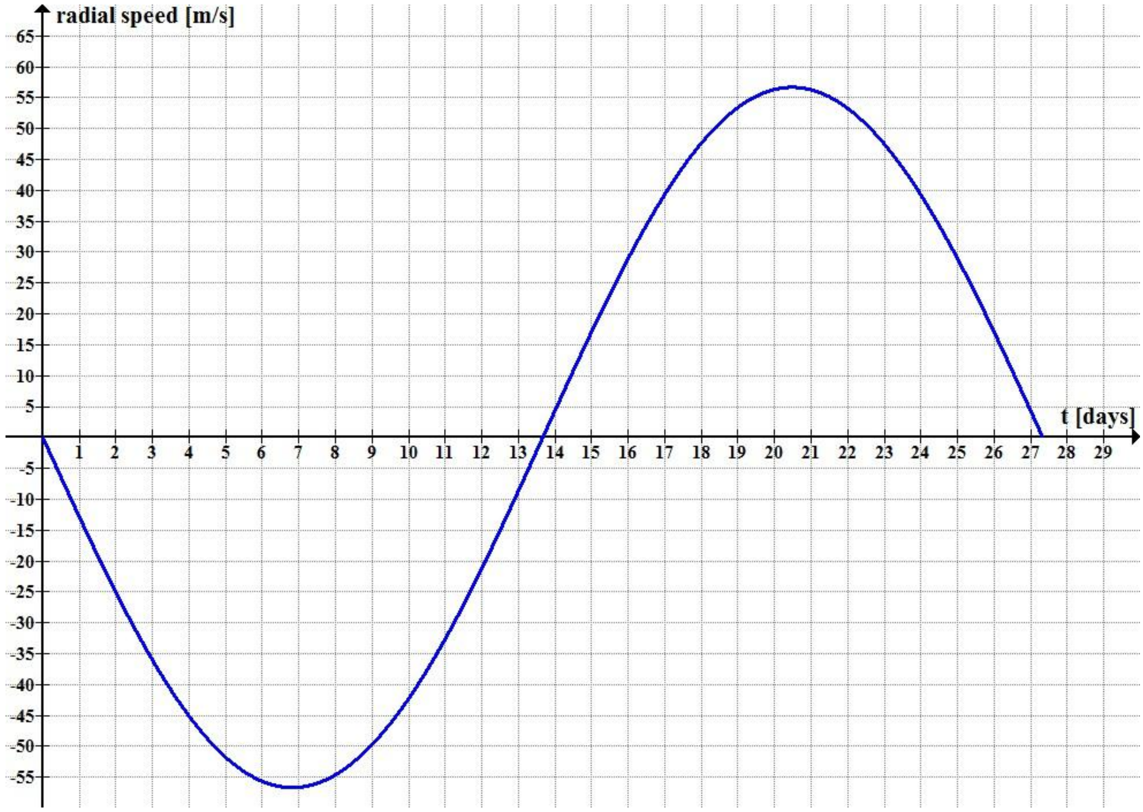

Figure 3: *Radial speed of the Moon due to the eccentricity of the orbit*

Note that $t = 0$ at apogee.

2 Calculation of the distance to the Moon

2.1 Introduction

We will calculate the distance to the reflector on the Moon based on a LLR measurement of the round trip time of a pulse. The calculation is very simplified, effects like the reduced speed of light in the atmosphere are ignored. The main point with the calculation is to inspect the difference between the SR and the ballistic theory of light.

In the following, let:

$\boldsymbol{t_f}$ be the transit time of the pulse from the observatory to the reflector on the Moon
$\boldsymbol{t_b}$ be the transit time of the pulse from the reflector on the Moon to the observatory
$\boldsymbol{t_r}$ be the round trip transit time of the pulse from observatory - reflector - observatory
$\boldsymbol{d_r}$ be the distance observatory-reflector at the time the pulse hits the reflector
$\boldsymbol{d_o}$ be the distance observatory-reflector at the time the pulse hits the observatory
$\boldsymbol{v}$ be the radial speed of the reflector as observed by the observatory
$\boldsymbol{c}$ be the speed of light in vacuum

In a LLR measurement it is the round trip transit time t_r that is measured, and the time when the pulse is detected at the observatory is registered. So what we want is the distance to the Moon d_o at the time the pulse is detected at the observatory, as a function of the round trip time.

$$d_o = f(t_r) \tag{11}$$

2.2 Calculation according to the Special Theory of Relativity

According to SR, the speed of light is isotropic in the momentarily comoving inertial frame of the observatory.

$$t_f = \frac{d_r}{c} \tag{12}$$

$$t_b = \frac{d_r}{c} \tag{13}$$

$$t_r = t_f + t_b = \frac{2}{c}\, d_r \tag{14}$$

$$d_r = \frac{c}{2}\, t_r \tag{15}$$

$$t_b = \frac{t_r}{2} \tag{16}$$

While the pulse was on it's way back, the reflector has moved a distance Δd:

$$\Delta d = v t_b = \frac{v}{2}\, t_r \tag{17}$$

$$d_o = d_r + \Delta d = \frac{c}{2}\left(1 + \frac{v}{c}\right) t_r \tag{18}$$

2.3 Calculation according to the ballistic theory of light.

According to the ballistic theory:

1. The energy of the light (observed in the mirror frame) is conserved at the reflection.
2. The Galilean transform applies.

The only possible consequence of this is that according to the ballistic theory, the speed of the reflected light in the mirror frame is equal to the speed of the incident light in same frame. Nothing else make sense.

In this case the speed of the incident light is $(c - v)$ and thus the speed of the reflected light is $(c - v)$, both as observed in the mirror frame.

Transformed to the observatory frame, the speed of the outgoing pulse is c, and the speed of the reflected pulse is $(c - 2v)$

Thus:

$$t_f = \frac{d_r}{c} \tag{19}$$

$$t_b = \frac{d_r}{c - 2v} \tag{20}$$

$$t_r = t_f + t_b = \frac{2\,(c - v)}{c\,(c - 2v)}\; d_r \tag{21}$$

$$d_r = \frac{c\,(c - 2v)}{2\,(c - v)}\; t_r \tag{22}$$

$$t_b = \frac{c}{2\,(c - v)}\; t_r \tag{23}$$

While the pulse was on it's way back, the reflector has moved a distance:

$$\Delta d = v t_b = \frac{vc}{2\,(c - v)}\; t_r \tag{24}$$

$$d_o = d_r + \Delta d = \frac{c\,(c - 2v)}{2\,(c - v)}\; t_r + \frac{vc}{2\,(c - v)}\; t_r = \frac{c}{2}\; t_r \tag{25}$$

2.4 Comparison between SR and the ballistic theory

If one particular measurement is analysed according to the ballistic theory and SR, the difference between the distance measurements will be $\left(1 - \frac{v}{c}\right)$. If v is in the order of 30 m/s (it is much bigger most of the time), the difference will be 40 m. We know that the LLR data analysed according to SR give a smooth orbit of the Moon. The data analysed according to the ballistic theory will give an impossible wobbly orbit of the Moon.

3 Conclusion

It is an irrefutable fact that the LLR falsifies the ballistic theory.