The influence of the curvature of space-time on LIDAR measurements of the distance to the Moon

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1 Introduction

The obvious frame of reference to use in a Lunar Laser Ranging (LLR) measurement of the distance to the Moon seems to be the *momentarily co-moving reference frame* (MCRF) of the observatory. This frame is however not a true inertial frame due to the curvature of space-time in the vicinity of the Earth. We will in the following investigate the influence of the curvature.

2 The scenario

Since we only are interested in the *difference* between a calculation of the distance in an inertial frame and in curved space time, we will use the following simplified scenario:
The observatory is on equator, and the reflector on the Moon is in the meridian at the celestial equator. We will ignore the rotation of the Earth and the motion of the Moon. A short laser pulse is emitted from the observatory, it bounces off a reflector on the Moon, and is detected at the observatory which measure the round trip time for the pulse.

Let R_e be the proper radius of the Earth, let R_m be be the proper distance from the centre of the Earth to the reflector on the Moon, and let $d = R_m - R_e$ be the proper distance from the observatory on the Earth to the reflector on the Moon. These proper distances are as they would be measured with metre sticks.
We will use the following values:

$$R_e = 6367446\ m, \quad R_m = 403958000\ m, \quad d = 397590554\ m \tag{1}$$

3 Calculation in an inertial frame

Calculated with the assumption that the MCRF is inertial the round trip time τ_r' measured by the observatory is:

$$\tau_r' = \frac{2d}{c} = 2.652438668087\ s \tag{2}$$

where $c = 2.99792458 \cdot 10^8\ m/s$, the speed of light in vacuum.

4 Calculation in the Schwarzschild frame

The Schwarzschild metric is:

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 \mathrm{d}t^2 - \frac{1}{\left(1 - \frac{2GM}{c^2 r}\right)} \mathrm{d}r^2 - r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\right) \tag{3}$$

where:

$\boldsymbol{s}$	is a space-time interval	
$\boldsymbol{t}$	is the Schwarzschild time coordinate	
$\boldsymbol{r}$	is the Schwarzschild radial coordinate	
$\boldsymbol{\theta}$	is the colatitude (angle from north)	
$\boldsymbol{\varphi}$	is the longitude	
$\boldsymbol{G}$	$= 6.67384 \cdot 10^{-11}\ Nm^2/kg^2$,	the gravitational constant
$\boldsymbol{M}$	$= 5.97258 \cdot 10^{24}\ kg$,	the mass of the Earth
$\boldsymbol{c}$	$= 2.99792458 \cdot 10^{8}\ m/s$,	the speed of light in vacuum

The Schwarzschild radius of the Earth:

$$r_s = \frac{2GM}{c^2} = 8.87006 \cdot 10^{-3}\ m \tag{4}$$

If we only consider radially space-time intervals, the equations becomes:

$$\mathrm{d}s^2 = \left(1 - \frac{r_s}{r}\right) c^2 \mathrm{d}t^2 - \frac{\mathrm{d}r^2}{\left(1 - \frac{r_s}{r}\right)} \tag{5}$$

The Schwarzschild radial coordinate r is not a proper distance, so we need to find the relation between the proper radial distance R and r. The proper radial distance along a radius from r_s to r is:

$$s = \int_{r_s}^{r} \frac{\mathrm{d}r}{\sqrt{1 - \frac{r_s}{r}}} \tag{6}$$

So:

$$R = r_s + \int_{r_s}^{r} \frac{\mathrm{d}r}{\sqrt{1 - \frac{r_s}{r}}} = r_s + \left[r\sqrt{1 - \frac{r_s}{r}} + \frac{r_s}{2} \ln\left(2r\left(\sqrt{1 - \frac{r_s}{r}} + 1\right) - r_s\right)\right]_{r_s}^{r} \tag{7}$$

$$R = r_s + \sqrt{r\,(r - r_s)} + r_s \ln\left(\sqrt{\frac{r}{r_s} - 1} + \sqrt{\frac{r}{r_s}}\right) \tag{8}$$

We have now found the proper radial distance as a function of the Schwarzschild radial coordinate $R(r)$. We would like to find the inverse $r(R)$, but this is extremely complex to find analytically. We can however find $r_e = r(R_e)$ and $r_m = r(R_m)$ numerically with the aid of a computer.

Using the values in (1) yields:

$$r_e = R_e - 0.101\ m = 6367445.899\ m \tag{9}$$

$$r_m = R_m - 0.119\ m = 403957999.881\ m \tag{10}$$

Light moves along a null-geodesic, and radially moving light will thus move according to the equation:

$$ds^2 = \left(1 - \frac{r_s}{r}\right)\ c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)} = 0 \tag{11}$$

$$dt^2 = \frac{dr^2}{c^2\left(1 - \frac{r_s}{r}\right)^2} \tag{12}$$

The round trip time for the pulse to go from the observatory to the reflector on the Moon and back is:

$$t_r = \int_{r_e}^{r_m} \frac{dr}{c\left(1 - \frac{r_s}{r}\right)} + \int_{r_m}^{r_e} -\frac{dr}{c\left(1 - \frac{r_s}{r}\right)} = \frac{2}{c}\left[r + \ln\left(r - r_s\right)\right]_{r_e}^{r_m} \tag{13}$$

$$= \frac{2}{c}\left(\left(r_m - r_e\right) + \ln\left(\frac{r_m - r_s}{r_e - r_s}\right)\right) = 2.65243866821\ s \tag{14}$$

This is the round trip time in the Schwarzschild time coordinate. To find the proper time τ_r measured by a clock at the observatory we use equation (5) with $ds = c\ d\tau$, $r = r_e$ and $dr = 0$:

$$d\tau^2 = \left(1 - \frac{r_s}{r_e}\right) dt^2 \tag{15}$$

$$\tau_r = \int_0^{t_r} \sqrt{1 - \frac{r_s}{r_e}}\ dt = \sqrt{1 - \frac{r_s}{r_e}}\ t_r = 2.652438668210\ s \tag{16}$$

5 Conclusion

Calculated in an inertial frame, the round trip time is (2):

$$\tau_r' = 2.652438668087\ s$$

and calculated in the Schwarzschild frame it is (16):

$$\tau_r = 2.65243866821\ s$$

So the extra delay due to the curvature of space-time is:

$$\Delta\tau_r = \tau_r - \tau_r' = 1.23 \cdot 10^{-10}\ s = 0.123\ ns \tag{17}$$

With no correction for the curvature of space-time, the error in the distance estimate would be:

$$\Delta d = \frac{\Delta\tau_r c}{2} = 0.0184\ m = 18.4\ mm \tag{18}$$

Not very much, but since the LLR project aims for a precision in the sub cm range, it must be accounted for.