Is the speed of light invariant according to relativity?

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1 Introduction

The answer to the question in the title is obvious for all reasonably knowledgeable persons. But not all the contributors to the Usenet group *sci.physics.relativity* belong to said class of people, as the following quote from a posting in this group illustrates.

Ralph Malcom Rabbidge, who is posting under the pseudonym Henry Wilson, wrote:

> " *Do you admit then that everything you have said about the GPS system is wrong? Are you not aware that your pet theory says light moves at c in the Earth's ECI frame and not the frame of the observatory?* "

The context of the quoted statement was a discussion about the Lunar Laser Ranging project, and the poster's idea seems to be that according to relativity, the speed of light can not be c both in *the Earth centred inertial reference frame*(ECI) and in *the momentarily comoving reference frame (MCRF) of the observatory.*[1] Or more generally: the speed of light can not at the same time be c in two inertial frames in relative motion.

We will in the following demonstrate that the round trip transit time of a light pulse in a LIDAR measurement can be calculated in two different inertial frames in relative motion, and will give the same answer if we assume that the speed of light is invariant.

[1] Neither of these reference frames are inertial in the strictest sense, because space-time is curved in the vicinity of the Earth. The speed of light will however be locally invariant in both frames.

2 Calculation of the round trip transit time in a LIDAR experiment

2.1 The scenario

A short laser pulse is emitted from a source and is reflected off a reflector and going back to a detector co-located with the source. The reflector is moving at the speed u relative to the source/detector. A clock at the source/detector measures the round trip time t_r.

The events of interest are:

$\boldsymbol{E_0}$: the pulse is emitted from the source
$\boldsymbol{E_1}$: the pulse hits the reflector
$\boldsymbol{E_2}$: the pulse is back at the source

We will in the following analyse the scenario in two inertial frames which are moving at the speed v relative to each other.

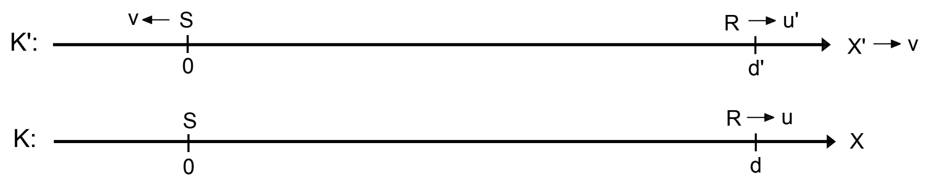

Figure 1: *The frames of reference*

2.2 Calculation in the rest frame of the source

Frame K is the rest frame of the source/detector S which is at the position $x = 0$ in K. The pulse is emitted at the time $t = 0$ and hits the reflector R when it is at the position $x = d$. Since the speed of light is c in frame K, the pulse will hit the reflector R at the time $t = \frac{d}{c}$, and will hit the source/detector S at the time $t = \frac{2d}{c}$. The coordinates of the events in K will thus be:

$$E_0: \qquad t_0 = 0 \qquad\qquad x_0 = 0 \tag{1}$$

$$E_1: \qquad t_1 = \frac{d}{c} \qquad\qquad x_1 = d \tag{2}$$

$$E_2: \qquad t_2 = \frac{2d}{c} \qquad\qquad x_2 = 0 \tag{3}$$

The round trip transit time is:

$$t_r = t_2 - t_0 = \frac{2d}{c} \tag{4}$$

Note that t_r is a proper time since it is measured with a single clock at S. t_r is the invariant space-time interval between the events E_0 and E_2.

2.3 Calculation in a reference frame where the source is moving

The inertial frame K' is moving at the speed v along the positive x axis of K, and their origins are aligned at $t' = t = 0$. The pulse is emitted at the time $t' = 0$ and hits the reflector R when it is at the position $x' = d'$. The source/detector S is moving at the speed v along the negative x' axis. Since the speed of light is c in frame K', the pulse will hit the reflector R at the time $t' = \frac{d'}{c}$. To find when the pulse will hit the source/detector S we must solve the equation: $t'c = 2d' + t'v$, which yields: $t' = \frac{2d'}{c-v}$. The position of S in K' when it is hit by the pulse will be $x' = -t'v = \frac{2d'v}{c-v}$.

The coordinates of the events in K' will thus be:

$$E_0: \quad t'_0 = 0 \qquad x'_0 = 0 \tag{5}$$

$$E_1: \quad t'_1 = \frac{d'}{c} \qquad x'_1 = d' \tag{6}$$

$$E_2: \quad t'_2 = \frac{2d'}{c-v} \qquad x'_2 = -\frac{2d'v}{c-v} \tag{7}$$

The round trip transit time is:

$$t'_r = t'_2 - t'_0 = \frac{2d'}{c-v} \tag{8}$$

Note that t'_r is not a proper time because it is read off two different clocks, one at $x' = 0$ and one at $x' = -\frac{2d'v}{c-v}$. We can however find the proper time measured by the clock at S as the invariant space-time interval between E_0 and E_2:

$$t_r^2 = (t'_2 - t'_0)^2 - \frac{(x'_2 - x'_0)^2}{c^2} = \left(\frac{2d'}{c}\right)^2 \cdot \frac{1 + \frac{v}{c}}{1 - \frac{v}{c}} \tag{9}$$

$$t_r = \frac{2d'}{c}\sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} \tag{10}$$

We can find the relationship between d and d' by transforming the spatial coordinate of E_1 from K to K' :

$$x'_1 = \frac{x_1 - vt_1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{d - \frac{dv}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{11}$$

$$d' = d\sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} \tag{12}$$

Combining (10) and (12) yields:

$$t_r = \frac{2d}{c} \tag{13}$$

Which is the same as (4).

3 Conclusion

When the round trip time of a LIDAR pulse measured by a clock on the source/detector is calculated in two different inertial reference frames, the result will be the same when it is assumed that the speed of light is invariant.