An Elementary Introduction to Centripetal Acceleration

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Velocity is a vector, it has a size and a direction. The size is often called the *speed*. In the case when an object is moving in a circular orbit, the *speed* is constant, but the *direction* is constantly changing, see fig.1. At the time t the velocity $\vec{v}\,(t)$ is in one direction, at the later time $(t + \Delta t)$, the velocity $\vec{v}\,(t + \Delta t)$ is in a different direction.

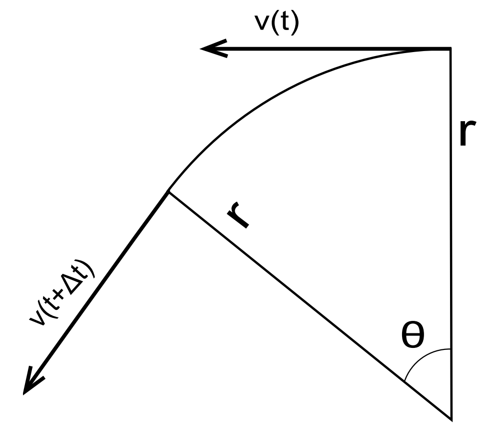

Figure 1: *The velocity at the time t and $(t + \Delta t)$*

The acceleration $\vec{a}$ is the change of the velocity with respect to time, that is the temporal derivative of the velocity.

$$\vec{a}\,(t) = \frac{\mathrm{d}}{\mathrm{d}t}\vec{v}\,(t) = \lim_{\Delta t \to 0} \frac{\vec{v}\,(t + \Delta t) - \vec{v}\,(t)}{\Delta t} \tag{1}$$

From fig.1 we can see that the object has moved an angle θ during the time Δt. Thus:

$$\Delta t = \frac{\theta}{\omega} = \frac{r\theta}{v} \tag{2}$$

where:

- $\boldsymbol{r}$ is the radius of the orbit
- $\boldsymbol{v}$ is the orbital speed of the object, $v = |\vec{v}|$
- $\boldsymbol{\omega}$ is the angular velocity of the object, $\omega = \frac{v}{r}$

The change of the velocity during the time Δt is:

$$\Delta\vec{v}\,(\Delta t) = \vec{v}\,(t + \Delta t) - \vec{v}\,(t) \tag{3}$$

From fig.1 we can see that θ is the angle between $\vec{v}\,(t)$ and $\vec{v}\,(t + \Delta t)$, and from fig. 2 we can see that φ is the angle between $\vec{v}\,(t)$ and $\Delta\vec{v}\,(\Delta t)$.

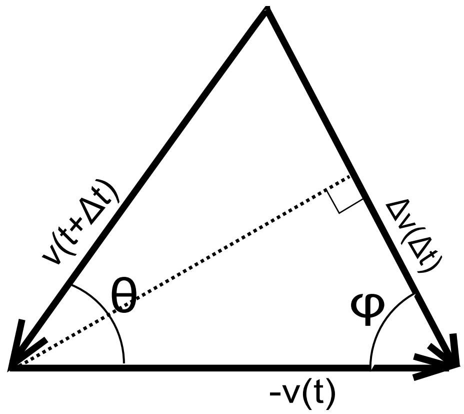

Figure 2: $\Delta\vec{v}\,(\Delta t)$

From fig.2 we have: $\varphi = \left(\frac{\pi}{2} - \frac{\theta}{2}\right)$, and from equation (2) we can see that $\theta \to 0$ when $\Delta t \to 0$.
Thus we have:

$$\angle\vec{a}\,(t) - \angle\vec{v}\,(t) = \lim_{\Delta t \to 0} \varphi = \lim_{\theta \to 0} \left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \frac{\pi}{2} \tag{4}$$

This means that the acceleration is always perpendicular to the velocity, it is pointing towards the centre of the circle and is therefore called *centripetal acceleration*.

Let us now find the size of the acceleration $|\vec{a}\,(t)|$. From fig.2 we can see:

$$|\Delta\vec{v}\,(\Delta t)| = 2v \sin\left(\frac{\theta}{2}\right) \tag{5}$$

From equation (1), (2) and (5) we get:

$$|\vec{a}\,(t)| = \lim_{\Delta t \to 0} \frac{|\vec{v}\,(t + \Delta t) - \vec{v}\,(t)|}{\Delta t} = \lim_{\Delta t \to 0} \frac{|\Delta\vec{v}\,(\Delta t)|}{\Delta t} = \lim_{\theta \to 0} \frac{2v \sin\left(\frac{\theta}{2}\right)}{\frac{r\theta}{v}} = \frac{v^2}{r} \cdot \lim_{\theta \to 0} \frac{\sin\left(\frac{\theta}{2}\right)}{\frac{\theta}{2}} = \frac{v^2}{r}$$

Summing up:

$$\begin{aligned} |\vec{v}\,(t)| &= v \\ \angle\vec{v}\,(t) &= \omega t \\ |\vec{a}\,(t)| &= \frac{v^2}{r} \\ \angle\vec{a}\,(t) &= \omega t + \frac{\pi}{2} \end{aligned}$$