Answer to Edgar L. Owen

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September 26, 2019

Edgar L. Owen's question

September 24, 2019 Edgar L. Owen wrote in the usenet group sci.physics.relativity:

Disregarding gravitation:

Twins A and B separate from any possible point in the universe, each take any separate path, then meet again at any possible location.

In general when they meet their clocks will read different elapsed proper times.

Is it true that in all possible cases each twin's elapsed proper time will be equal to the sum of its Lorentz time dilation due to its velocity RELATIVE to the eventual meeting point? Where the Lorentz time dilation is of their own proper time RELATIVE to a clock at the eventual meeting point (i.e. the coordinate time of their clock as measured relative to a clock at the eventual meeting point)?

If not what is the simple rule that determines the difference in proper times in the most general case above?

Answer to Edgar

How to find the proper time of an object (like a twin)

You can choose any inertial frame $\mathcal{K}$ with the metric:

$$(c\ \mathrm{d}\tau)^2 = (c\ \mathrm{d}t)^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \tag{1}$$

where τ is the proper time of the object, while $[t, x, y, z]$ are the coordinates of the inertial frame of reference, and c is the speed of light in vacuum.

If we use the coordinate time t as parameter, the equation can be written:

$$\mathrm{d}\tau^2 = \left(1 - \frac{1}{c^2}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right]\right)\mathrm{d}t^2 \tag{2}$$

Which can be simplified to:

$$\mathrm{d}\tau = \sqrt{1 - \frac{v\,(t)^2}{c^2}}\ \ \mathrm{d}t \tag{3}$$

where $\vec{v}\left(t\right) \underset{\mathcal{K}}{\rightarrow} \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right)$ and $v\left(t\right)^2 = \vec{v}\left(t\right) \cdot \vec{v}\left(t\right) = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2$.

The proper time between event $E_0\left[t_0, x_0, y_0, z_0\right]$ and event $E_0\left[t_1, x_1, y_1, z_1\right]$ on the object's world line will be:

$$\tau = \int_{t_0}^{t_1} \sqrt{1 - \frac{v\left(t\right)^2}{c^2}}\ \mathrm{d}t \tag{4}$$

Note that $v\left(t\right)$ may be any function of t.

The proper time will now be compared to the coordinate time of the chosen inertial frame, that is, it will be given as $k\left(t_1 - t_0\right)$, so if t is measured in seconds, so will τ be.
Since the temporal coordinates of E_0 and E_1, t_0 and t_1, are frame dependent, τ may appear to be the same. However, if you choose a different inertial frame, $v\left(t\right)$ will also change so τ will be invariant.

Since the spacetime interval between E_0 and E_1 is time-like, it will always be possible to choose an inertial frame where the spatial coordinates of E_0 and E_1 are equal. This will be the most 'natural' frame to choose. In this case $\left(t_1 - t_0\right)$ will be a proper time, which is a measure of the invariant spacetime interval between E_0 and E_1.

(The invariant spacetime interval between E_0 and E_1 will be: $s^2 = -c^2\left(t_1 - t_0\right)^2$.)

The twins scenario

The twins A and B are co-located at the event E_0, when they start and travel along separate paths in spacetime to the event E_1, when they meet again.

The "the simple rule that determines the difference in proper times in the most general case" is:

$$\tau_A - \tau_B = \int_{t_0}^{t_1} \sqrt{1 - \frac{v_A\left(t\right)^2}{c^2}}\ \mathrm{d}t \quad - \int_{t_0}^{t_1} \sqrt{1 - \frac{v_B\left(t\right)^2}{c^2}}\ \mathrm{d}t \tag{5}$$

Where τ_A and $v_A\left(t\right)$ are the proper time and speed of twin A, and τ_B and $v_B\left(t\right)$ are ditto for twin B.

A simple example

The scenario

The twins A and B are co-located at the event E_0, when they start and travel along separate paths in spacetime to the event E_1, when they meet again. A travels at constant speed to event E_A, when she turns around and travels at constant speed to E_1, while B travels at constant speed to event E_B, when she turns around and travels at constant speed to E_1

Solution in the inertial frame of reference $\mathbf{K_1}$ where $\mathbf{E_0}$ and $\mathbf{E_1}$ are co-located

Let the coordinates of $E_0 \underset{K_1}{\rightarrow} (0,0,0,0)$ and $E_1 \underset{K_1}{\rightarrow} (t_1,0,0,0)$.

Let twin A travel at the speed v to the event $E_A \underset{K_1}{\rightarrow} \left(\frac{t_1}{2}, L, 0, 0\right)$, where she turns abruptly around and travels back to E_1. So $t_1 = \frac{2L}{v}$.

Let twin B travel at the speed $2v$ to the event $E_B \underset{K_1}{\rightarrow} \left(\frac{t_1}{2}, -2L, 0, 0\right)$, where she turns abruptly around and travels back to E_1. So $t_1 = \frac{2L}{v}$.

$$E_0 \underset{K_1}{\rightarrow} (0,0,0,0) \tag{6}$$

$$E_1 \underset{K_1}{\rightarrow} \left(\frac{2L}{v}, 0, 0, 0\right) \tag{7}$$

$$E_A \underset{K_1}{\rightarrow} \left(\frac{L}{v}, L, 0, 0\right) \tag{8}$$

$$E_B \underset{K_1}{\rightarrow} \left(\frac{L}{v}, -2L, 0, 0\right) \tag{9}$$

$$v_A(t) = v \text{ for } t \le \frac{L}{v} \tag{10}$$

$$v_A(t) = -v \text{ for } \frac{L}{v} < t \le \frac{2L}{v} \tag{11}$$

$$v_A(t)^2 = v^2 \text{ for } 0 < t \le \frac{2L}{v} \tag{12}$$

$$\tau_A = \int_0^{\frac{2L}{v}} \sqrt{1 - \frac{v^2}{c^2}}\ \mathrm{d}t = \frac{2L}{v}\sqrt{1 - \frac{v^2}{c^2}} \tag{13}$$

$$v_B(t) = -2v \text{ for } t \le \frac{L}{v} \tag{14}$$

$$v_B(t) = 2v \text{ for } \frac{L}{v} < t \le \frac{2L}{v} \tag{15}$$

$$v_B(t)^2 = 4v^2 \text{ for } 0 < t \le \frac{2L}{v} \tag{16}$$

$$\tau_B = \int_0^{\frac{2L}{v}} \sqrt{1 - \frac{4v^2}{c^2}}\ \mathrm{d}t = \frac{2L}{v}\sqrt{1 - \frac{4v^2}{c^2}} \tag{17}$$

$$\tau_A - \tau_B = \frac{2L}{v}\left(\sqrt{1 - \frac{v^2}{c^2}} - \sqrt{1 - \frac{4v^2}{c^2}}\right) \tag{18}$$

Solution in another inertial frame of reference $\mathbf{K_2}$

Let us use an inertial frame which is moving at the speed v relative to the frame in the previous section. Lorentz transformation of the coordinates of the events will then give:

$$E_0 \underset{K_2}{\rightarrow} (0,0,0,0) \tag{19}$$

$$E_1 \underset{K_2}{\rightarrow} \left(\frac{\frac{2L}{v}}{\sqrt{1 - \frac{v^2}{c^2}}}, -\frac{2L}{\sqrt{1 - \frac{v^2}{c^2}}}, 0, 0\right) \tag{20}$$

$$E_A \underset{K_2}{\rightarrow} \left(\frac{L}{v}\sqrt{1 - \frac{v^2}{c^2}}, 0, 0, 0\right) \tag{21}$$

$$E_B \underset{K_2}{\rightarrow} \left(\frac{L}{v}\frac{\left(1 + 2\frac{v^2}{c^2}\right)}{\sqrt{1 - \frac{v^2}{c^2}}}, -\frac{3L}{\sqrt{1 - \frac{v^2}{c^2}}}, 0, 0\right) \tag{22}$$

$$v_A(t) = 0 \text{ for } 0 < t \le \frac{L}{v}\sqrt{1 - \frac{v^2}{c^2}} \tag{23}$$

$$v_A(t) = \frac{-2v}{1 + \frac{v^2}{c^2}} \text{ for } \frac{L}{v}\sqrt{1 - \frac{v^2}{c^2}} < t \le \frac{\frac{2L}{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{24}$$

$$\tau_A = \int_0^{\frac{L}{v}\sqrt{1-\frac{v^2}{c^2}}} \mathrm{d}t + \int_{\frac{L}{v}\sqrt{1-\frac{v^2}{c^2}}}^{\frac{\frac{2L}{v}}{\sqrt{1-\frac{v^2}{c^2}}}} \sqrt{1-\left(\frac{-2\frac{v}{c}}{1+\frac{v^2}{c^2}}\right)^2} \mathrm{d}t = \frac{L}{v}\sqrt{1-\frac{v^2}{c^2}} + \frac{L}{v}\sqrt{1-\frac{v^2}{c^2}} \tag{25}$$

$$\tau_A = \frac{2L}{v}\sqrt{1-\frac{v^2}{c^2}}$$

$$v_B(t) = \frac{-3v}{1+2\frac{v^2}{c^2}} \text{ for } 0 < t \leq \frac{\frac{L}{v}\left(1+2\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} \tag{26}$$

$$v_B(t) = \frac{v}{1-2\frac{v^2}{c^2}} \text{ for } \frac{\frac{L}{v}\left(1+2\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} < t \leq \frac{\frac{2L}{v}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{27}$$

$$\begin{aligned}
\tau_B &= \int_0^{\frac{\frac{L}{v}\left(1+2\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}}} \sqrt{1-\left(\frac{-3\frac{v}{c}}{1+2\frac{v^2}{c^2}}\right)^2} \mathrm{d}t + \int_{\frac{\frac{L}{v}\left(1+2\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}}}^{\frac{\frac{2L}{v}}{\sqrt{1-\frac{v^2}{c^2}}}} \sqrt{1-\left(\frac{\frac{v}{c}}{1-2\frac{v^2}{c^2}}\right)^2} \mathrm{d}t \\
&= \frac{L}{v}\sqrt{1-\left(\frac{-3\frac{v}{c}}{1+2\frac{v^2}{c^2}}\right)^2} \cdot \frac{\left(1+2\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{L}{v}\sqrt{1-\left(\frac{\frac{v}{c}}{1-2\frac{v^2}{c^2}}\right)^2} \cdot \frac{\left(1-2\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} \\
&= \frac{L}{v}\sqrt{1-4\frac{v^2}{c^2}} + \frac{L}{v}\sqrt{1-4\frac{v^2}{c^2}} \\
\tau_B &= \frac{2L}{v}\sqrt{1-4\frac{v^2}{c^2}}
\end{aligned} \tag{28}$$

$$\tau_A - \tau_B = \frac{2L}{v}\left(\sqrt{1-\frac{v^2}{c^2}} - \sqrt{1-\frac{4v^2}{c^2}}\right) \tag{29}$$

Compare this to (18).

Note that v is a specific speed, and L is a specific length which are defined in K_1. They are constants which have no special significance in K_2. They are used in K_2 to make it obvious that τ_A and τ_B are the same when calculated in two different inertial frames, and thus are invariant.