# Simulation of the solar system according to GR

Paul B. Andersen

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## 1 Numeric values for the mean elements of the planets at Epoch J2000

The orbital elements of the planets are:

| | |
|---|---|
| Semi major axis | $\boldsymbol{a}$ |
| Eccentricity | $\boldsymbol{e}$ |
| Inclination to ecliptic | $\boldsymbol{i}$ |
| Mean longitude | $\boldsymbol{\lambda}$ |
| Longitude of perihelion | $\boldsymbol{\varpi}$ |
| Longitude of ascending node | $\boldsymbol{\Omega}$ |

Elements calculated from the above:

| | | |
|---|---|---|
| Perihelion distance | $\boldsymbol{a_p}$ | $= a\,(1-e)$ |
| Aphelion distance | $\boldsymbol{a_a}$ | $= a\,(1+e)$ |
| Argument of perihelion | $\boldsymbol{\omega}$ | $= \varpi - \Omega$ |
| Mean anomaly | $\boldsymbol{M}$ | $= \lambda - \varpi$ |
| True anomaly | $\boldsymbol{T}$ | $= M + \left(2e - \frac{e^3}{4}\right)\sin(M) + \frac{5e^2}{4}\sin(2M) + \frac{13e^3}{12}\sin(3M)$ |

The numerical values of the orbital elements are from:

**J.L. Simon & al:**
Numerical expressions for precession formulae and mean elements for the Moon and planets

Solar mass: $m_{sun} = 1.98847 \cdot 10^{30}$ kg. Mass of planets are inclusive mass of moons.

**Mercury**

$$m = (\frac{m_{sun}}{6023600}) = 3.301132 \cdot 10^{23}\text{kg}$$
$$a = 0.3870983098 \text{ AU} = 5.7909082898 \cdot 10^{10}\text{m}$$
$$\lambda = 252.25090552^o$$
$$e = 0.2056317526$$
$$\varpi = 77.45611904^o$$
$$i = 7.00498625^o$$
$$\Omega = 48.33089304^o$$

**Venus**

$$m = (\frac{m_{sun}}{408523.71}) = 4.867453 \cdot 10^{24}\text{kg}$$
$$a = 0.7233298200 \text{ AU} = 10.820860089 \cdot 10^{10}\text{m}$$
$$\lambda = 181.97980085^o$$
$$e = 0.0067719164$$
$$\varpi = 131.56370300^o$$
$$i = 3.39466189^o$$
$$\Omega = 76.67992019^o$$

**Earth**

$$m = (\frac{m_{sun}}{328900.56}) = 6.045809 \cdot 10^{24}\text{kg}$$
$$a = 1.0000010178 \text{ AU} = 14.959802296 \cdot 10^{10}\text{m}$$
$$\lambda = 100.46645683^o$$
$$e = 0.0167086342$$
$$\varpi = 102.93734808^o$$
$$i = 0.0^o$$
$$\Omega = 174.87317577^o$$

**Mars**

$$m = (\frac{m_{sun}}{3098708}) = 6.417094 \cdot 10^{24}\text{kg}$$
$$a = 1.5236793419 \text{ AU} = 22.793918518 \cdot 10^{10}\text{m}$$
$$\lambda = 355.43299958^o$$
$$e = 0.0934006477$$
$$\varpi = 336.06023394^o$$
$$i = 1.84972648^o$$
$$\Omega = 49.55809321^o$$

**Jupiter**

$$m = (\frac{m_{sun}}{1047.3486}) = 1.898575 \cdot 10^{27}\text{kg}$$
$$a = 5.2026032092 \text{ AU} = 77.829836219 \cdot 10^{10}\text{m}$$
$$\lambda = 34.35151874^o$$
$$e = 0.0484979255$$
$$\overline{\omega} = 14.33120687^o$$
$$i = 1.30326698^o$$
$$\Omega = 100.46440702^o$$

**Saturn**

$$m = (\frac{m_{sun}}{3497.90}) = 5.684754 \cdot 10^{26}\text{kg}$$
$$a = 9.5549091915 \text{ AU} = 142.939406978 \cdot 10^{10}\text{m}$$
$$\lambda = 50.07744430^o$$
$$e = 0.0555481426$$
$$\overline{\omega} = 93.05723748^o$$
$$i = 2.48887878^o$$
$$\Omega = 113.66550252^o$$

**Uranus**

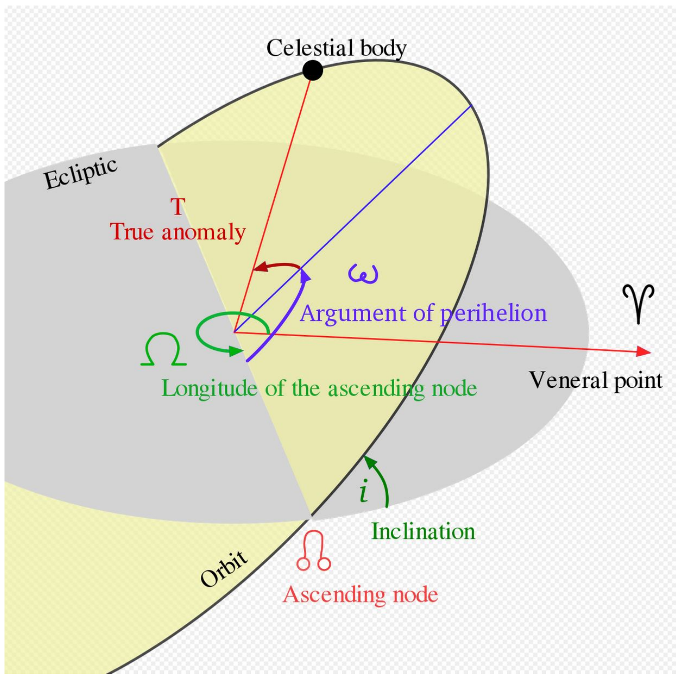
$$m = (\frac{m_{sun}}{22902.94}) = 8.682160 \cdot 10^{25}\text{kg}$$
$$a = 19.2184460618 \text{ AU} = 287.503860901 \cdot 10^{10}\text{m}$$
$$\lambda = 314.05500511^o$$
$$e = 0.0463812221$$
$$\overline{\omega} = 173.00529106^o$$
$$i = 0.77319689^o$$
$$\Omega = 74.00595701^o$$

**Neptune**

$$m = (\frac{m_{sun}}{19412.24}) = 1.024338 \cdot 10^{26}\text{kg}$$
$$a = 30.1103868694 \text{ AU} = 450.444976162 \cdot 10^{10}\text{m}$$
$$\lambda = 304.34866548^o$$
$$e = 0.0094557470$$
$$\overline{\omega} = 48.12027554^o$$
$$i = 1.76995259^o$$
$$\Omega = 131.78405702^o$$

The used data for the planets are:

| | | |
|---|---|---|
| The mass | $\boldsymbol{m}$ | where $m$ includes the mass of the planet's moons |
| The orbital period | $\boldsymbol{P}$ | |
| The semi major axis | $\boldsymbol{a}$ | half the distance between the planet at perihelion and aphelion |
| The perihelion distance | $\boldsymbol{a_p}$ | distance Sun - planet at perihelion |
| The inclination | $\boldsymbol{i}$ | to the ecliptic plane |
| The mean anomaly | $\boldsymbol{M}$ | at Epoch J2000 |
| Longitude of ascending node | $\boldsymbol{\Omega}$ | |
| Argument of perihelion | $\boldsymbol{\omega}$ | |




| **Planets** | **Mercury** | **Venus** | **Earth** | **Mars** | **Jupiter** | **Saturn** | **Uranus** | **Neptun** |
|---|---|---|---|---|---|---|---|---|
| $\boldsymbol{m}$ $[\boldsymbol{kg}]$ | 3.301132E23 | 4.867453E24 | 6.045809E24 | 6.417094E24 | 1.898575E27 | 5.684754E26 | 8.682160E25 | 1.024338E26 |
| $\boldsymbol{P}$ $[\boldsymbol{days}]$ | 87.969116 | 224.7016 | 365.256363 | 686.9816 | 4332.589 | 10759.22 | 30685.4 | 60189.0 |
| $\boldsymbol{a}$ $[\mathbf{10^{10}}\boldsymbol{m}]$ | 5.7909082898 | 10.820860089 | 14.959802296 | 22.793918518 | 77.829836219 | 142.93940698 | 287.50386090 | 450.44497616 |
| $\boldsymbol{a_p}$ $[\mathbf{10^{10}}\boldsymbol{m}]$ | 4.6001136690 | 10.747582129 | 14.709844432 | 20.664951765 | 74.055250620 | 134.99938841 | 274.16908047 | 446.18568243 |
| $\boldsymbol{i}$ $[\ ^o]$ | $7.00499^o$ | $3.39466^o$ | $0.00000^o$ | $1.84973^o$ | $1.30327^o$ | $2.48888^o$ | $0.77320^o$ | $1.76995^o$ |
| $\boldsymbol{M}$ $[\ ^o]$ | $174.796^o$ | $50.115^o$ | $358.617^o$ | $19.3870^o$ | $20.020^o$ | $317.020^o$ | $142.2386^o$ | $256.228^o$ |
| $\boldsymbol{\Omega}$ $[\ ^o]$ | $48.331^o$ | $76.680^o$ | $-11.26064^o$ | $49.558^o$ | $49.558^o$ | $113.665^o$ | $74.006^o$ | $131.784^o$ |
| $\boldsymbol{\omega}$ $[\ ^o]$ | $29.124^o$ | $54.884^o$ | $114.20783^o$ | $286.502^o$ | $273.867^o$ | $339.392^o$ | $96.999^o$ | $276.336^o$ |

Table 1: Planet data at Epoch J2000

Solar mass $= 1.98847 \cdot 10^{30}$ kg.

## 2 Calculation of the initial positions and velocities of the planets at Epoch J2000

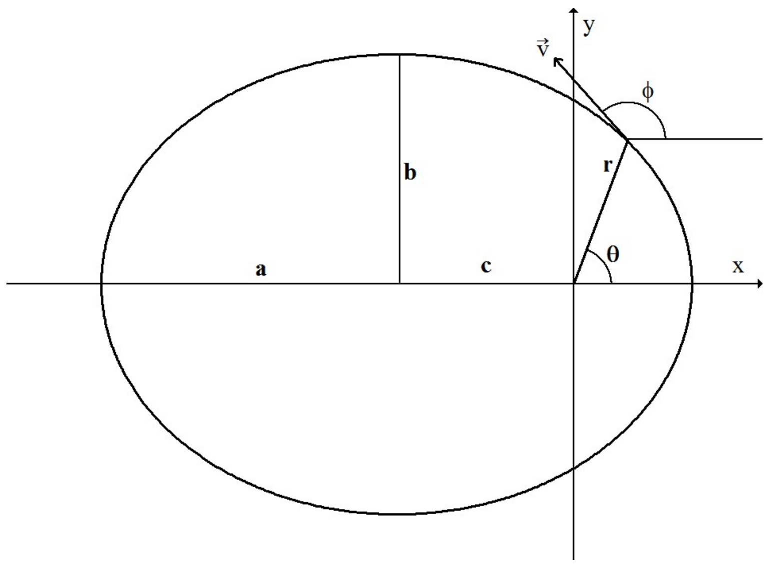



Figure 1:

With reference to fig. 2, we have generally:

$r(\theta) = \dfrac{a\left(1-\epsilon^2\right)}{1+\epsilon\cos\theta}$, $|\vec{v}| = \sqrt{G\left(m_s+m\right)\left(\dfrac{2}{r}-\dfrac{1}{a}\right)}$ and $\phi = \arctan\left(-\dfrac{\cos\theta+\epsilon}{\sin\theta}\right)$
where $G$ is the gravitational constant , $m_s$ is the mass of the Sun and $m$ is the mass of the planet.

From these data, the following data can be calculated:

The eccentricity $\epsilon = 1 - \dfrac{a_p}{a}$

The distance Sun - planet at aphelion $a_a = a\left(1+\epsilon\right)$

The true anomaly at the reference time Epoch J2000 is:
$T = M + \left(2\epsilon - \dfrac{\epsilon^3}{4}\right)\sin\left(M\right) + \dfrac{5\epsilon^2}{4}\sin\left(2M\right) + \dfrac{13\epsilon^3}{12}\sin\left(3M\right)$

## 2.1 The planets' positions and velocities in the solar frame with the orbital plane as reference plane and the axis Sun-perihelion as reference direction

The position vector in the solar frame with the orbital plane as reference plane and the axis Sun-perihelion as as reference direction is, in spherical coordinates, magnitude $r$, longitude $\Theta$ and latitude $\varphi$:

$$\vec{r_1} = \vec{r}\,(r, \Theta, \varphi) \text{ where } r = \frac{a\left(1-\epsilon^2\right)}{1+\epsilon\cos T},\ \Theta = T \text{ and } \varphi = 0 \tag{1}$$

The velocity of the planet, in spherical coordinates, magnitude $v$, longitude $\Theta$ and latitude $\varphi$, is:

$$\vec{v_1} = \vec{v}\,(v, \Theta, \varphi) \text{ where } v = \sqrt{G\left(m_s+m\right)\left(\frac{2}{r}-\frac{1}{a}\right)}, \Theta = \arctan\left(-\frac{\cos T+\epsilon}{\sin T}\right) \text{ and } \varphi = 0 \tag{2}$$

where G is the gravitational constant, $m_s$ is the mass of the Sun, and $r = \dfrac{a\left(1-\epsilon^2\right)}{1+\epsilon\cos T}$.

## 2.2 The planets' positions and velocities in the solar frame with the orbital plane as reference plane and Aries as reference direction

We will however have the reference direction to Aries, so with reference to fig. 1, we must rotate the vectors above by the angle $\Omega + \omega$.

So in the solar frame, with the orbital plane as reference plane and Aries as reference direction, the position vector is:

$$\vec{r_2} = \vec{r}\,(r, \Theta, \varphi) \text{ where } r = \frac{a\left(1-\epsilon^2\right)}{1+\epsilon\cos T},\ \Theta = (T+\Omega+\omega) \text{ and } \varphi = 0 \tag{3}$$

and the initial velocity is:

$$\vec{v_2} = \vec{v}\,(v, \Theta, \varphi) \text{ where } v = \sqrt{G\left(m_s+m\right)\left(\frac{2}{r}-\frac{1}{a}\right)}, \Theta = \arctan\left(-\frac{\cos T+\epsilon}{\sin T}\right) + \Omega + \omega \text{ and } \varphi = 0 \tag{4}$$

where G is the gravitational constant, $m_s$ is the mass of the Sun, and $r = \dfrac{a\left(1-\epsilon^2\right)}{1+\epsilon\cos T}$.

## 2.3 The planets' positions and velocities in the solar frame with the ecliptic plane as reference plane and Aries as reference direction

The position vector $\vec{r_2}$ and velocity $\vec{v_2}$ are in the x-y plane. But since the orbital plane has an angle $i$ to the ecliptic plane, and the ecliptic plane now shall be the x-y plane, the position vector and velocity must be rotated an angle $i$ around the axis through the ascending node, that is around an axis with the angle $\Omega$ from Aries.

Let the function $Tilt$ be this operation.

$$\vec{r_0} = Tilt\left(\vec{r_2}, \Omega, i\right) \tag{5}$$
$$\vec{v_0} = Tilt\left(\vec{v_2}, \Omega, i\right) \tag{6}$$

We now have the initial positions and the initial velocities for all the planets in the solar frame with the ecliptic plane as reference plane, and Aries as reference direction, $\vec{r_{0i}}$ and $\vec{v_{0i}}$, $i = 1 \rightarrow 8$.

## 2.4 The planets' positions and velocities in the center of mass frame with the ecliptic plane as reference plane and Aries as reference direction

Let $\vec{r_{00}}$ and $\vec{v_{00}}$ be two zero vectors representing the position and velocity of the Sun in the solar frame.

We will now transform these to the center of mass frame. We will first find the centre of mass in the solar frame:

$$\vec{R} = \frac{\sum_{i=0}^{8} m_i \cdot \vec{r_{0i}}}{\sum_{i=0}^{8} m_i} \tag{7}$$

And the velocity of the centre of mass in the solar frame:

$$\vec{V_r} = \frac{\sum_{i=0}^{8} m_i \cdot \vec{v_{0i}}}{\sum_{i=0}^{8} m_i} \tag{8}$$

In the following we will let $\vec{P_i}(t)$ where $i = 0 \to 8$ be the position of the Sun ($i = 0$) and planets ($i = 1 \to 8$) at the time $t$, and $\vec{V_i(t)}$ the velocity of the Sun and planets, both in the in frame of reference where the centre of mass is stationary, the reference plane is the ecliptic plane, and the reference direction is Aries.

We then have:

$$\vec{P_i}(0) = \vec{r_{0i}} - \vec{R} \tag{9}$$

$$\vec{V_i}(0) = \vec{v_{0i}} - \vec{V_r} \tag{10}$$

Where the time $t = 0$ is Epoch J2000.

# 3 Simulation of the Solar System

When we know the initial positions and the initial velocities of the Sun and planets in the centre of gravity frame (CG-frame), we can find the position and velocity of a body at any time $n \cdot \Delta t$, where $\Delta t$ is a time increment:

$$\vec{V_i}(n \cdot \Delta t) = \vec{V_i}((n-1) \cdot \Delta t) + \Delta t \cdot \vec{A_i}((n-1) \cdot \Delta t) \tag{11}$$

$$\vec{P_i}(n \cdot \Delta t) = \vec{P_i}((n-1) \cdot \Delta t) + \Delta t \cdot \vec{V_i}(n \cdot \Delta t) \tag{12}$$

where $\vec{A_i}$ is the gravitational acceleration of body $i$ caused by all the other bodies in the Solar system.

The gravitational acceleration of a body in the solar system predicted by GR is very complex, see:

**Theodore D. Moyer:**
Formulation for Observed and Computed Values of Deep Space Network Data Types for Navigation

Equation (4-26) with $\beta = \gamma = 1$ gives the post-Newtonian approximation of the gravitational acceleration predicted by GR.

We can however simplify the problem somewhat by using the GR acceleration for the Sun's acceleration of a planet, and for the planets' acceleration of the Sun, while we use the Newtonian acceleration of the planets' acceleration of each other.

The post-Newtonian approximation of the gravitational acceleration predicted by GR is according to:
**Barman Shahid-Saless1 and Donald K. Yeomans:**
RELATIVISTIC EFFECTS ON THE MOTION OF ASTEROIDS AND COMETS

See equation (3.11):

$$\frac{\mathrm{d}^2\vec{r}}{c^2\mathrm{d}t^2} = -\frac{\mu}{r^3}\vec{r} + \frac{\mu}{r^3}\left[\left(4\frac{\mu}{r} - \frac{v^2}{c^2}\right)\vec{r} + 4\frac{(\vec{r}\cdot\vec{v})\,\vec{v}}{c^2}\right] \tag{13}$$

With $\mu = \frac{GM}{c^2}$ we get:

$$\frac{\mathrm{d}^2\vec{r}}{c^2\mathrm{d}t^2} = -\frac{GM}{r^3c^2}\vec{r} + \frac{GM}{r^3c^2}\left[\left(4\frac{GM}{rc^2} - \frac{v^2}{c^2}\right)\vec{r} + 4\frac{(\vec{r}\cdot\vec{v})\,\vec{v}}{c^2}\right] \tag{14}$$

The Sun's gravitational acceleration of a planet is according to GR:

$$\vec{a_p} = \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = -\frac{GM}{r^2}\left(1 - 4\frac{GM}{rc^2} + \frac{v^2}{c^2}\right)\hat{r} + \frac{4GMv^2\,(\hat{r}\cdot\hat{v})}{r^2c^2}\cdot\hat{v} \tag{15}$$

The acceleration of the Sun by a planet is:

$$\vec{a_s} = -\frac{m}{M}\vec{a_p} \tag{16}$$

Where:

| | | |
|---|---|---|
| $\vec{a_p}$ | = Sun's acceleration of a planet | $[m/s^2]$ |
| $\vec{a_s}$ | = planet's acceleration of Sun | $[m/s^2]$ |
| $\vec{r}$ | = distance vector sun - planet | $[m]$ |
| $r$ | = $\lvert\vec{r}\rvert$, magnitude of $\vec{r}$ | $[m]$ |
| $\hat{r}$ | = unity vector parallel to $\vec{r}$ | |
| $\vec{v}$ | = velocity of planet in solar frame | $[m/s]$ |
| $v$ | = $\lvert\vec{v}\rvert$, magnitude of $\vec{v}$ | $[m/s]$ |
| $\hat{v}$ | = unity vector parallel to $\vec{v}$ | |
| $c$ | = speed of light | $[m/s]$ |
| $G$ | = Gravitational constant | $[m^3\cdot kg^{-1}\cdot s^{-2}]$ |
| $M$ | = solar mass | $[kg]$ |
| $m$ | = planet mass | $[kg]$ |

We define the acceleration of the planet $i$ to be: $\vec{A}_i = \vec{A}_{si} + \vec{A}_{pi}$, where $\vec{A}_{si}$ is the acceleration caused by the Sun while $\vec{A}_{pi}$ is the acceleration caused by all the other planets.

$$\vec{A}_{si}\,(n\cdot\Delta t) = -\frac{GM}{r^2}\left(1 - 4\frac{GM}{rc^2} + \frac{v^2}{c^2}\right)\hat{r} + \frac{4GMv^2\,(\hat{r}\cdot\hat{v})}{r^2c^2}\cdot\hat{v} \tag{17}$$

Where:

$$r = |\vec{P}_i\,((n-1)\cdot\Delta t) - \vec{P}_0\,((n-1)\cdot\Delta t)\,|$$
$$\hat{r} = \frac{\vec{P}_i\,((n-1)\cdot\Delta t) - \vec{P}_0\,((n-1)\cdot\Delta t)}{|\vec{P}_i\,((n-1)\cdot\Delta t) - \vec{P}_0\,((n-1)\cdot\Delta t)\,|}$$
$$v = |\vec{V}_i\,((n-1)\cdot\Delta t) + \vec{V}_0\,((n-1)\cdot\Delta t)\,|$$
$$\hat{v} = \frac{\vec{V}_i\,((n-1)\cdot\Delta t) + \vec{V}_0\,((n-1)\cdot\Delta t)}{|\vec{V}_i\,((n-1)\cdot\Delta t) + \vec{V}_0\,((n-1)\cdot\Delta t)\,|}$$

$$\vec{A}_{pi}\left(n\cdot\Delta t\right)=\sum_{k=1}^{8}\frac{\delta(i,k)\cdot Gm_k}{\left(\vec{P}_k\left((n-1)\cdot\Delta t\right)-\vec{P}_i\left((n-1)\cdot\Delta t\right)\right)^2}\cdot\frac{\vec{P}_k\left((n-1)\cdot\Delta t\right)-\vec{P}_i\left((n-1)\cdot\Delta t\right)}{\left|\vec{P}_k\left((n-1)\cdot\Delta t\right)-\vec{P}_i\left((n-1)\cdot\Delta t\right)\right|} \quad (18)$$

Where $m_k$ is the mass of planet $k$ and $\delta(i,k)=1$ when $i\neq k$ and $\delta(i,k)=0$ when $i=k$.

The total acceleration of planet $i$ is:

$$\vec{A}_i\left(n\cdot\Delta t\right)=\vec{A}_{si}\left(n\cdot\Delta t\right)+\vec{A}_{pi}\left(n\cdot\Delta t\right) \quad (19)$$

The acceleration of the Sun is:

$$\vec{A}_0\left(n\cdot\Delta t\right)=-\sum_{k=1}^{8}\frac{m_k}{m_0}\cdot\vec{A}_{sk}\left(n\cdot\Delta t\right) \quad (20)$$